math.AG/0001106
HUB-EP-00/03
TUW–00/01

Reflexive polyhedra, weights and toric Calabi–Yau fibrations

Maximilian KREUZER*

Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8–10, A-1040 Wien, AUSTRIA

and

Harald SKARKE#

Humboldt Universität zu Berlin, Institut für Physik, QFT,
Invalidenstraße 110, D-10115 Berlin, GERMANY

ABSTRACT

During the last years we have generated a large number of data related to Calabi–Yau hypersurfaces in toric varieties which can be described by reflexive polyhedra. We classified all reflexive polyhedra in three dimensions leading to K3 hypersurfaces and have nearly completed the four dimensional case relevant to Calabi–Yau three-folds. In addition, we have analysed for many of the resulting spaces whether they allow fibration structures of the types that are relevant in the context of superstring dualities. In this survey we want to give background information both on how we obtained these data, which can be found at our web site, and on how they may be used. We give a complete exposition of our classification algorithm at a mathematical (rather than algorithmic) level. We also describe how fibration structures manifest themselves in terms of toric diagrams and how we managed to find the respective data. Both for our classification scheme and for simple descriptions of fibration structures the concept of weight systems plays an important role.

HUB-EP-00/03
TUW–00/01
January 2000

*e-mail: kreuzer@hep.itp.tuwien.ac.at
#e-mail: skarke@physik.hu-berlin.de

1 Introduction

A few years after Calabi–Yau manifolds had found their way into physics it was conjectured that they should actually come in pairs with opposite Euler number, since an exchange of complex structure and Kähler moduli in physics corresponds to a change of sign in the definition of the charge, or, equivalently, an exchange of particles and anti-particles [1, 2]. This phenomenon is called mirror symmetry. Although the situation is complicated by the fact that there are rigid Calabi–Yau manifolds whose "mirror string compactifications" do not have a straightforward geometrical interpretation [3, 4], the search for the mirror manifolds proved to be an extremely fruitful enterprise from both the physicists' and the mathematicians' perspective [5–7].

The first systematic constructions of large classes of Calabi–Yau threefolds as complete intersections in products of projective spaces [8] did not seem to support the mirror hypothesis because the resulting manifolds all had negative Euler numbers. But when the attention was extended to weighted projective spaces, it turned out that the blow up parameters of the quotient singularities can provide large positive contributions. The first substantial list of pairs of Hodge numbers resulting from constructions of this type [9] was almost mirror symmetric in the sense that only for a few percent of the Hodge data the respective mirror pair was not in the list. A complete classification [10, 11], however, made the picture worse, and abelian quotients [12], which make a subclass of these spaces perfectly symmetric [13, 14], did not help with this problem either.

Batyrev's construction of toric Calabi–Yau hypersurfaces [15], which is manifestly mirror symmetric while generalizing the above results, provided a solution to this puzzle. In this framework the geometrical data is encoded by a reflexive polyhedron, i.e. a lattice polyhedron whose facets are all at distance 1 from the unique interior point (see below). Toric geometry turned out to provide a very efficient tool for the analysis of many physical aspects of Calabi–Yau compactifications, including the physics of perturbative [16] and non-perturbative [17–20] topology changing transitions, as well as fibration structures that are important in string dualities [21, 22].

This made a constructive classification of reflexive polyhedra a useful and interesting enterprise. Our approach to this problem [23] was partly inspired by our experience with the classification of weighted projective spaces that admit transversal quasi-homogeneous polynomials [24]. Indeed, as it turned out, the Newton polyhedra that correspond to polynomials

defining CY hypersurfaces in weighted $\mathbb{P}^4$ are all reflexive [25,26] and provide a canonical resolution of the ambient space singularities (this is no longer true in higher dimensions). Actually, regardless of the transversality condition, a diophantine equation of the form $\sum n_i a_i = d$ with positive coefficients n_i, $d = \sum n_i$, and with the set of solutions restricted to $a_i \geq 0$ gives a simple way to produce lattice polyhedra with at most one interior point (this is a necessary condition for reflexivity): We may regard this as an embedding of the lattice into a higher dimensional space with the polyhedron being contained in the finite intersection of an affine subspace with the non-negative half-spaces. All lattice points, except for the candidate interior point, whose coordinates are all equal to 1, are located on some coordinate hyperplane $a_i = 0$.

We may then ask ourselves if all reflexive polyhedra are contained in polyhedra that can be embedded in this way. In the next section we will show that the answer is assertive provided that we allow for an embedding with higher codimension $k - n$, i.e. we also consider solutions to more than one equation of the above form,

$$\sum_{i=1}^{k} a_i n_i^{(j)} = d^{(j)}, \qquad d^{(j)} = \sum_{i=1}^{k} n_i^{(j)}, \qquad j = 1, \ldots, k-n, \tag{1}$$

but with some of the coefficients $n_i^{(j)}$ equal to zero according to a certain pattern. What then makes our construction work is the fact that there is only a finite set of coefficients that lead to lattice polytopes with an interior point. The collections $n_i^{(j)}$ of non-negative numbers are called weight systems in the case of a single equation and combined weight systems if $k - n > 1$. If we shift our coordinates to $x_i = a_i - 1$ the resulting polyhedron lies in a linear subspace of the embedding space determined by $\sum_i n_i^{(j)} x_i = 0$, is bounded by $x_i \geq -1$ and has the origin of the embedding space as its interior point. These linear coordinates are more useful for many general considerations wheras the affine coordinates a_i are better suited for quickly finding the lattice points in a given example.

Let us illustrate with the example in *fig.* 1 how we can obtain a weight system for a given reflexive polyhedron Δ with vertices in some n-dimensional lattice M. Reflexivity implies that the dual (or polar) polytope Δ^* defined in eq. (4) below has its vertices on the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$. In our case Δ^* is already minimal in the sense that we lose the interior point (IP) if we drop any of its vertices and take the convex hull of the remaining vertices. The set of vertices of $\nabla = \Delta^*$ can be decomposed into the two triangles (V_1, V_2, V_5) and (V_3, V_4, V_5) that both contain the IP in their lower dimensional interior. As we will show later, similar decompositions are always possible for minimal polyhedra. For both triangles the barycentric

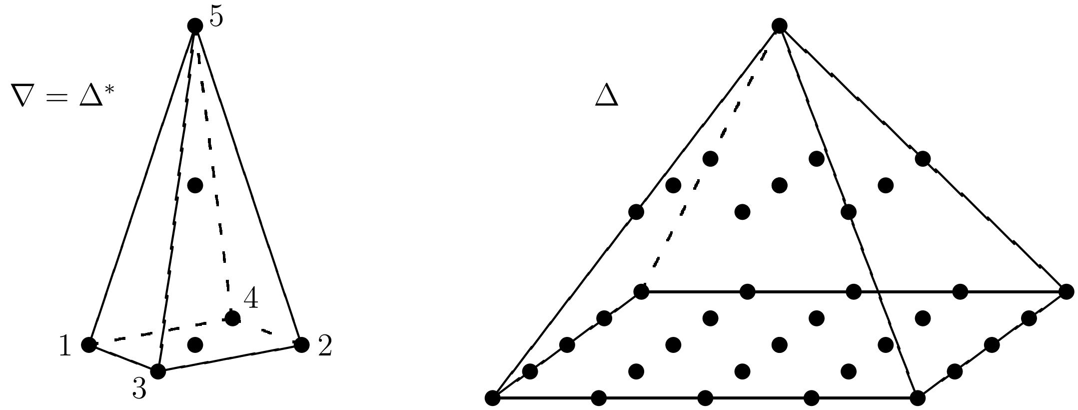

Fig. 1: A minimal polyhedron ∇ that corresponds to a combined weight system.

coordinates of the IP are given by $\mathbf{q} = (1/4, 1/4, 1/2)$, i.e. $\sum q_i V_i = 0$ and $\sum q_i = 1$, where the sum is over the indices of the vertices for any of the two triangles. Rescaling the coefficients to integers $n_i^{(j)} = d^{(j)} q_i^{(j)}$ we arrive at the weight system $n_i^{(1)} = (1,1,0,0,2)$, $n_i^{(2)} = (0,0,1,1,2)$. We will demonstrate in the next section that weights obtained in this way can always be used to describe the dual polytope Δ as in eq. 1. In the present case, this construction leads to

$$x_1 + x_2 \qquad\qquad + 2x_5 = 0, \tag{2}$$

$$x_3 + x_4 + 2x_5 = 0. \tag{3}$$

Eliminating, for example, x_2 and x_4 it is easily checked that we indeed reconstructed Δ (note that $a_i = x_i + 1$ is the lattice distance of a point from the facet dual to V_i). If we keep all points with $x_i \geq -1$ then, in our example, Δ^* is equal to ∇. In general Δ^* will not be minimal and we first have to drop some vertices of Δ^* to arrive at a minimal polytope ∇ whose simplex decomposition leads to a weight system. If we drop points from Δ in such a way that $\Delta' \subset \Delta$ is reflexive, then Δ'^* becomes larger. The vertices of ∇ remain vertices of $\Delta'^* \supset \nabla$ as long as the bounding hyperplanes $x_i = -1$, which in our case support all facets of Δ, are affinely spanned by facets of Δ'.

A different way to generate a 'smaller' Δ is to keep the vertices but to go to a coarser M lattice: We may, for example, demand $x_1 - x_3 \in 2\mathbb{Z}$ or $x_1 + x_5 \in 2\mathbb{Z}$. Correspondingly, the N lattice becomes finer and is no longer generated by the vertices of ∇. In general there will occur additional lattice points in ∇. The coarsest lattice that keeps all vertices of Δ and the IP is obtained by imposing $x_1 - x_3 \in 4\mathbb{Z}$ and $x_1 + x_5 \in 2\mathbb{Z}$. Actually, in our example, this

exchanges ∇ and Δ.

In practice, because of the huge number of solutions, an enumeration of all reflexive polyhedra seems to be possible only in up to 4 dimensions. This leads to a further simplification of the procedure because in up to 4 dimensions all polytopes Δ that correspond to a minimal ∇ are reflexive [26]. Moreover, Δ is contained in a larger polytope $\hat{\Delta}$ if and only if $\hat{\Delta}^*$ is contained in Δ^*. Therefore only minimal polytopes for which Δ^* does not contain any reflexive subpolytope are necessary ingredients for our classification scheme. We will show that in 4 dimension there are 308 reflexive polytopes that contain all others as subpolytopes, provided that we also consider sublattices. Finding all relevant lattices is a subtle point and our strategy to solve this problem will be described below. There are at least 25 additional maximal reflexive polytopes that can be obtained from these 308 objects on sublattices.

While one of the main insights of the 'first superstring revolution' was the fact that Calabi–Yau spaces are crucial for string compactifications, it was found during the 'second string revolution' that fibration structures of Calabi–Yau manifolds are essential for understanding various non-perturbative string dualities. In particular, K3 fibrations are required for the duality between heterotic and IIA theories [27, 28] and elliptic fibrations are needed for F-theory compactifications [22,29,30]. Again toric geometry provides beautiful tools for studying the respective structures. As we will see, the polytope Δ_{f}^* corresponding to the fiber manifests itself as a subpolytope of Δ^* with the same interior point, whereas the base space is a toric variety whose fan can be determined by projecting the original fan along the linear subspace spanned by Δ_{f}^*. While we never attempted to give a complete classification of structures of this type, we did create large lists of fibration structures [31, 32].

Our data are accessible at our web site [33], and we plan to make the source code of our programs available in the near future. Since one of the motivations for writing this contribution was to give useful background material for anyone interested in applying our data, we would like to briefly mention some older results on our web page that will not be discussed in the remainder of this paper. These are mostly related to weighted projective spaces and, in the physical context, to Landau-Ginzburg models [2, 34]. We classified all 10839 weight systems allowing transversal quasihomogeneous polynomials [10, 24] with singularity index 3, leading to Landau-Ginzburg models with a central charge of c=9 and computed the corresponding numbers of (anti) chiral states in the superconformal field theories (this includes the 7555 transversal weights for weighted $\mathbb{P}^4$). Vafa's formulas for these numbers [34] inspired the

definition of what Batyrev et al. call string theoretic Hodge numbers [35]. We also extended these results to arbitrary abelian quotients that leave a transversal polynomial invariant [12] (and included the modifications by discrete torsions [36], which correspond to topologically non-trivial background 2-form fields in the physical context [37]). Since the Newton polyhedra are reflexive also for abelian quotients, the resulting Hodge numbers (without discrete torsion) are all recovered in the toric context. Nevertheless our results might be useful when working in weighted projective spaces, since transversal polynomials in general have larger symmetries than the complete Newton polyhedra.

We will not discuss Calabi–Yau data obtained by other groups here. An important class of spaces that we did not consider consists of complete intersection Calabi–Yau varieties. The classification of these objects in products of projective spaces was given in [8], and Klemm has produced a sizeable list of codimension two complete intersections in weighted projective spaces which is accessible via internet [38]. Work on toric complete intersections and nef partitions [39–41] is in progress. Further web pages with relevant information are [42, 43].

In the next section we give a self-contained exposition of our classification algorithm and of the results in 3 and 4 dimensions. In section 3 we discuss the implications of these results for the geometry of toric K3 and Calabi–Yau hypersurfaces. In section 4 we explain the toric realization of fibrations where both the fibered space and the fiber have vanishing first Chern classes. We discuss how weight systems can be used to encode such fibrations and how this is related to fibrations in weighted projective spaces. We also provide an appendix with several tables that summarise some of our results.

2 Classification of Reflexive Polyhedra

In this section we give a self-contained exposition of our methods and results on the classification of reflexive polyhedra, without reference to toric geometry. Nevertheless, as we will see in the next section, some of the concepts used here, in particular the concept of weights, have interpretations in terms of geometry.

A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$, and for our present purposes a polyhedron is the same thing as a polytope (in particular, it is always bounded, which need not be true if a polyhedron is defined as the intersection of a finite number of half

spaces).

We will be interested in the case where we have a pair of lattices $M \simeq \mathbb{Z}^n$ and $N = \mathrm{Hom}(M, \mathbb{Z}) \simeq \mathbb{Z}^n$ and their real extensions $M_\mathbb{R} \simeq \mathbb{R}^n$ and $N_\mathbb{R} \simeq \mathbb{R}^n$. A polyhedron $\Delta \subset M_\mathbb{R}$ is called a lattice (or integer) polyhedron if the vertices of Δ lie in M.

Definition: A polytope $\Delta \subset \mathbb{R}^n$ has the 'interior point property' or 'IP property', if $\mathbf{0}$ (the origin of $\mathbb{R}^n$) is in the interior. A simplex with this property is an IP simplex.

Definition: For any set $\Delta \subset M_\mathbb{R}$ the dual (or polar) set $\Delta^* \subset N_\mathbb{R} = M_\mathbb{R}^*$ is given by

$$\Delta^* = \{y \in N_\mathbb{R} : \quad \langle y, x \rangle \geq -1 \quad \forall x \in \Delta\}, \tag{4}$$

where $\langle y, x \rangle$ is the duality pairing between $y \in N_\mathbb{R}$ and $x \in M_\mathbb{R}$.

If Δ is a polytope with the IP property, then Δ^* is also a polytope with the IP property and $(\Delta^*)^* = \Delta$.

Definition: A lattice polyhedron $\Delta \subset M_\mathbb{R}$ is called reflexive if its dual $\Delta^* \subset N_\mathbb{R}$ is a lattice polyhedron w.r.t. the lattice N dual to M.

The main idea of our classification scheme is to construct a set of polyhedra such that every reflexive polyhedron is a subpolyhedron of one of the polyhedra in this set. By duality, every reflexive polyhedron must contain one of the duals of these polyhedra, so we are looking for polyhedra that are minimal in some sense. In the following subsection we will give a definition of minimality that depends only on the way in which a polytope is spanned by its vertices, without reference to a lattice or details of the linear structure. We will see that this allows for a very rough classification with only a few objects in low dimensions. The corresponding characterisation of polyhedra can be refined by specifying explicitly the linear relations between the vertices with the help of weight systems. We will see that these weight systems can be used in a simple way to find the polyhedra dual to the minimal ones and to check whether they can possibly contain reflexive polyhedra; the main criterion here is the existence of a dual pair of lattices such that a minimal polytope is a lattice polyhedron and the convex hull of the lattice points of the dual has the IP property. The classification of the relevant weight systems leads to a finite number of polytopes that contain all reflexive polytopes, with the subtlety that only the linear structure but not the lattice on which some polytope may be reflexive is specified. In the final subsection we solve this problem by showing how to identify all lattices on which a polyhedron given in terms of its linear structure can be reflexive, and present the results of

our classification scheme.

2.1 Minimal polyhedra and their structures

We will later give various definitions of minimality, each of which has advantages and disadvantages. Here we define the weakest form of minimality, but the one that is most useful, where we forget for the time being about the lattice structure and concentrate on the vertex structure only.

Definition: A minimal polyhedron $\nabla \subset \mathbb{R}^n$ is defined by the following properties:
1. ∇ has the IP property.
2. If we remove one of the vertices of ∇, the convex hull of the remaining vertices of ∇ does not have the IP property.

Obviously every polytope $\nabla \subset \mathbb{R}^n$ with the IP property contains at least one minimal polytope spanned by a subset of the vertices of ∇. Before asking ourselves which minimal polytopes can be subpolytopes of reflexive polyhedra, we will now analyse the possible general structures of minimal polytopes.

Lemma 1: A minimal polytope $\nabla \subset \mathbb{R}^n$ with vertices $V_1, \cdots, V_k$ is either a simplex or contains an n'-dimensional minimal polytope $\nabla' := \mathrm{ConvexHull}\{V_1, \cdots, V_{k'}\}$ and an IP simplex $S := \mathrm{ConvexHull}(R \cup \{V_{k'+1}, \cdots, V_k\})$ with $R \subset \{V_1, \cdots, V_{k'}\}$ such that $k - k' = n - n' + 1 \geq 2$ and $\dim S \leq n'$.
Proof: If ∇ is a simplex, there is nothing left to prove. Otherwise, we first note that every vertex V of ∇ must belong to at least one IP simplex: It is always possible to find a triangulation of ∇ such that every n-simplex in this triangulation has V as a vertex (just triangulate the cone whose apex is V and whose one dimensional rays are $V\tilde{V}$, where the $\tilde{V}$ are the other vertices of ∇). As $\mathbf{0}$ must belong to at least one of these simplices, it must lie on some simplicial face which then is an IP simplex. Now consider the set of all IP simplices consisting of vertices of ∇. Any subset of this set will define a lower dimensional minimal polytope: The fact that $\mathbf{0}$ is interior to each simplex means that it is a positive linear combination of the vertices of any such simplex, and therefore $\mathbf{0}$ can also be written as a positive linear combination of all vertices involved. If the corresponding polytope were not minimal, our original ∇ could not be minimal, either. Among all lower dimensional minimal polytopes, take one (call it ∇') with the maximal

dimension n' smaller than n. $\mathbb{R}^n$ factorizes into $\mathbb{R}^{n'}$ and $\mathbb{R}^n/\mathbb{R}^{n'} \cong \mathbb{R}^{n-n'}$ (equivalence classes in $\mathbb{R}^n$). The remaining vertices define a polytope $\nabla_{n-n'}$ in $\mathbb{R}^n/\mathbb{R}^{n'}$. If $\nabla_{n-n'}$ were not a simplex, it would contain a simplex of dimension smaller than $n-n'$ which would define, together with the vertices of ∇', a minimal polytope of dimension s with $n' < s < n$, in contradiction with our assumption. Therefore $\nabla_{n-n'}$ is a simplex. Because of minimality of ∇, each of the $n-n'+1$ vertices of $\nabla_{n-n'}$ can have only one representative in $\mathbb{R}^n$, implying $k-k' = n-n'+1$. The equivalence class of $\mathbf{0}$ can be described uniquely as a positive linear combination of these vertices. This linear combination defines a vector in $\mathbb{R}^{n'}$, which can be written as a negative linear combination of $\le n'$ linearly independent vertices of ∇'. These vertices, together with those of $\nabla_{n-n'}$, form the simplex S. By the maximality assumption about ∇', dim$S$ cannot exceed dim∇'. $\square$

Definition: For an n-dimensional minimal polytope ∇ with k vertices, an IP simplex structure is a collection of subsets $S_i, 1 \le i \le k-n$ of the set of vertices of ∇, such that:
The convex hull of the vertices in each S_i is an IP simplex,
$\nabla_j = \text{ConvexHull} \bigcup_{i=1}^{j} S_i$ is a lower dimensional minimal polytope for every $j \in \{1, \ldots, k-n\}$,
$\nabla_{k-n} = \nabla$ and
$S_j \setminus \bigcup_{i=1}^{j-1} S_i$ contains at least two vertices.

Corollary: Every minimal polytope allows an IP simplex structure.
Proof: If ∇ is a simplex, this is obvious. Otherwise one can choose $S_{k-n} = S$ and $\nabla_{k-n-1} = \nabla'$ with S and ∇' as in lemma 1 and proceed inductively. $\square$

Lemma 2: Denote by $\{S_i\}$ an IP simplex structure. Then $S_i - \bigcup_{j \ne i} S_j$ never contains exactly one point.
Proof: An IP simplex contains line segments VV' with $V' = -\varepsilon V$, where ε is a positive number. If a simplex $S = \text{ConvexHull}\{V_1, \cdots, V_{s+1}\}$ has all of its vertices except one (V_{s+1}) in common with other simplices, then all points in the linear span of S are nonnegative linear combinations of the V_j and the $-\varepsilon_j V_j$ with $j \le s$, thus showing that V_{s+1} violates the minimality of ∇. $\square$

The following example shows that an IP simplex structure need not be unique:

Example: $n = 5$, $\nabla = \text{ConvexHull}\{V_1, \cdots, V_8\}$ with

$$\begin{aligned} &V_1 = (1,1,0,0,0),\ V_2 = (1,-1,0,0,0),\ V_3 = (-1,0,1,0,0),\ V_4 = (-1,0,-1,0,0), \\ &V_5 = (-1,0,0,1,0),\ V_6 = (-1,0,0,-1,0),\ V_7 = (1,0,0,0,1),\ V_8 = (1,0,0,0,-1). \end{aligned} \tag{5}$$

∇ contains the IP simplices $S_{1234} = V_1V_2V_3V_4$ (in the $x_1x_2x_3$–plane), S_{1256} (in the $x_1x_2x_4$–plane), S_{3478} (in the $x_1x_3x_5$–plane), S_{5678} (in the $x_1x_4x_5$–plane) and the 4-dimensional minimal polytopes ∇_{123456}, ∇_{123478}, ∇_{125678}, ∇_{345678}. Any set of three of the four IP simplices defines an IP simplex structure.

Lemma 3: For dimensions $n = 1, 2, 3, 4$ of $\mathbb{R}^n$ precisely the following IP simplex structures of minimal polyhedra are possible:

$n = 1$: $\{S_1 = V_1V_2\}$;

$n = 2$: $\{S_1 = V_1V_2V_3\}$,
$\{S_1 = V_1V_2,\ S_2 = V_1'V_2',\}$;

$n = 3$: $\{S_1 = V_1V_2V_3V_4\}$,
$\{S_1 = V_1V_2V_3,\ S_2 = V_1'V_2'\}$,
$\{S_1 = V_1V_2V_3,\ S_2 = V_1V_2'V_3'\}$,
$\{S_1 = V_1V_2,\ S_2 = V_1'V_2',\ S_3 = V_1''V_2''\}$;

$n = 4$: As in the first column of table 1 in the appendix.

Proof: Recursive application of lemma 1 and use of lemma 2 shows that these are the only possible structures. Explicit realisations of these structures will be presented later. $\square$

2.2 Weight systems

Any IP polytope, and therefore any reflexive polyhedron, must obviously contain one of the minimal polyhedra encountered in the last subsection. The structures found there are rather coarse, so now we have to face the task of suitably refining them in such a way that they become useful for our goal of classifying reflexive polyhedra. In particular, we will find that the linear relations between the vertices of minimal polyhedra can be encoded by sets of real numbers called weight systems, and we will adress the question of which weight systems can occur if a minimal polyhedron is a subpolyhedron of some reflexive polytope.

The fact that a simplex spanned by vertices V_i contains the origin in its interior is equivalent to the condition that there exist positive real numbers (weights) q_i such that $\sum q_iV_i = 0$. As these numbers are unique up to a common factor, it is convenient to choose some normalization such as $\sum q_i = 1$.

Definition: A weight system is a collection of positive real numbers (weights) q_i with $\sum q_i = 1$.

A weight system corresponding to an IP simplex with vertices V_i is the normalized set of numbers q_i such that $\sum q_i V_i = 0$. A combined weight system (CWS) corresponding to a minimal polyhedron endowed with an IP simplex structure is the collection of weight systems $q_i^{(j)}$ corresponding to the IP simplices S_j occurring there, with $q_i^{(j)} = 0$ if $V_i \notin S_j$. We call a (combined) weight system rational if all of the q_i are rational numbers.

If a minimal polyhedron ∇ is a lattice polyhedron, a corresponding CWS will always be rational. In this case it is possible to normalise the weights as positive integers n_i with no common divisor; then $q_i = n_i/d$ with $d = \sum n_i$. We will use both conventions for describing weight systems. By the definition of a lattice polyhedron, any lattice on which a minimal polyhedron ∇ is integer must contain the lattice N_{coarsest} generated by the vertices of ∇.

Definition: Given a minimal polyhedron $\nabla \subset N_{\mathbb{R}}$, we define the lattice N_{coarsest} as the lattice in $N_{\mathbb{R}}$ generated linearly over $\mathbb{Z}$ by the vertices of ∇ and the lattice $M_{\text{finest}} \subset M_{\mathbb{R}}$ as the lattice dual to N_{coarsest}.

Lemma 4: If ∇ is a minimal polyhedron with vertices V_i and $\mathbf{q}$ a CWS corresponding to an IP simplex structure of ∇, then:
a) The map $M_{\mathbb{R}} \to \mathbb{R}^k$, $X \to \mathbf{x} = (x_1, \ldots, x_k)$ with $x_i = \langle V_i, X\rangle$ defines an embedding such that the image of $M_{\mathbb{R}}$ is the subspace defined by $\sum_i q_i^{(j)} x_i = 0 \ \forall j$.
b) ∇^* is isomorphic to the polyhedron defined in this subspace by $x_i \geq -1$ for $i = 1, \ldots, k$.
c) If $\mathbf{q}$ is rational, then M_{finest} is isomorphic to the sublattice of $\mathbb{Z}^k = \{(x_1, \ldots, x_k) \text{ integer}\} \subset \mathbb{R}^k$ determined by the equations $\sum_i q_i^{(j)} x_i = 0$.
Proof: a) $\sum q_i^{(j)} V_i = 0$ implies $\sum_i q_i^{(j)} x_i = 0$. Conversely, the x_i determine X because a point in $M_{\mathbb{R}}$ is uniquely determined by its duality pairings with a set of generators (here, the V_i) of the dual space.
b) follows from the form of the embedding map and the definition of the dual polytope (4).
c) If X belongs to any lattice M such that ∇ is integer on the dual lattice N, the corresponding x_i must be integer. If the x_i are integer, then X has integer pairings with the generators V_i of N_{coarsest}, so X belongs to M_{finest}. □

Corollary: An IP simplex structure together with the specification of a CWS uniquely determines a minimal polyhedron up to isomorphism.
Proof: By lemma 4, ∇^* and hence ∇ is uniquely determined by the CWS. □

As our example after lemma 2 shows, an IP simplex structure need not be unique, so it is possible that two different CWS may correspond to the same minimal polytope. In such a situation, the weight systems of one CWS must be linear combinations of those of the other CWS with coefficients that are not all nonnegative. Since all weights must be positive, this can only happen if there is an IP simplex such that all of its vertices also belong to other IP simplices in the same IP simplex structure. This can happen only for $n \geq 5$, as one can see by explicitly checking all cases for $n \leq 4$. Thus, for $n \geq 5$ it might be preferable to work with equivalence classes of CWS leading to the same minimal polytopes instead of using CWS only.

Definition: If $\mathbf{q}$ is a rational CWS corresponding to a minimal polyhedron ∇, we define $\Delta(\mathbf{q})$ as the convex hull of $\nabla^* \cap M_{\text{finest}}$. We say that $\mathbf{q}$ has the IP property if $\Delta(\mathbf{q})$ has the IP property.

Corollary: If a CWS has the IP property, then every single weight system occurring in it also has the IP property.

Proof: Without loss of generality we can assume that the single weight system is $\mathbf{q}^{(1)}$ with $q_i^{(1)} > 0$ for $i \leq l$ and $q_i^{(1)} = 0$ for $i > l$. There is a natural projection π from $\mathbb{Z}^k$ as in lemma 4 to $\mathbb{Z}^l$ by restriction to the first l coordinates. Our construction implies that the projection of the lattice polytope in $\mathbb{Z}^k$ is a subpolytope of the lattice polytope in $\mathbb{Z}^l$ determined by $\sum_i q_i^{(1)} x_i = 0$ and $x_i \geq -1$. If $\mathbf{0}^l = \pi(\mathbf{0}^k)$ were not in the interior of the polytope in $\mathbb{Z}^l$, then $\mathbf{0}^k$ could not be in the interior of the polytope in $\mathbb{Z}^k$. $\square$

Lemma 5: Let l denote the number of weights of a weight system. Then the following statements hold:

$l = 2$: There is a single IP weight system, namely $(1, 1)$.

$l = 3$: There are three IP weight systems, namely $(1, 1, 1)$, $(1, 1, 2)$ and $(1, 2, 3)$.

$l = 4$: There are the 95 IP weight systems shown in table 3.

$l = 5$: There are 184,026 IP weight systems which can be found at our web site [33].

Proof: The classification of IP weight systems is based on the study of which integer points are allowed by lemma 4. Assume that a weight system $q_1, \cdots, q_l$ allows a collection of points with coordinates $x_i \geq -1$ as in lemma 4, including the interior point with $x_i = 0\ \forall i$. If these points fulfill an equation of the type $\sum_{i=1}^l a_i x_i = 0$ with $\mathbf{a} \neq \mathbf{q}$, then the weight system must also allow at least one point with $\sum_{i=1}^l a_i x_i > 0$ and at least one point with $\sum_{i=1}^l a_i x_i < 0$ to ensure that $\mathbf{0}$ is really in the interior. The latter inequality is the one that we actually use for the algorithm:

Starting with the point $\mathbf{0}$, we see that unless our weight system is $\mathbf{q} = (1/l, \cdots, 1/l)$, there must be at least one point with $\sum_{i=1}^{l} x_i < 0$. For $l \leq 5$ there are only a few possibilities, and after choosing some point $\mathbf{x}_1$, we can look for some simple equation fulfilled by $\mathbf{0}$ and $\mathbf{x}_1$ and proceed in the same way.

If $l = 2$, any weight system except $(1/2, 1/2)$ would have to allow an integer point with $x_1+x_2 < 0$, $x_1 \geq -1$ and $x_2 \geq -1$. Such a point has no positive coordinate and therefore cannot be allowed by a (positive) weight system.

For $l = 3$ the classification is still easily carried out by hand: Unless $\mathbf{q} = (1/3, 1/3, 1/3)$, we need at least one point with $x_1 + x_2 + x_3 < 0$. As points where no coordinate is greater than 0 would be in conflict with the positivity of the weight system, we need the point $(1, -1, -1)$ (up to a permutation of indices). Now we note that $\mathbf{0}$ and $(1, -1, -1)$ both fulfill $2x_1 + x_2 + x_3 = 0$, so $\mathbf{q} = (1/2, 1/4, 1/4)$ or we need a point with $2x_1+x_2+x_3 < 0$. The only point allowed by this inequality which leads to a sensible weight system is $(-1, 2, -1)$, leading to $\mathbf{q} = (1/2, 1/3, 1/6)$.

For $l = 4$ and $l = 5$ we have implemented this strategy in a computer program that produced 99 and 200653 candidates for IP weight systems, respectively. Finally, explicit constructions of $\Delta(\mathbf{q})$ show that four of the 99 weight systems with $l = 4$ and 16627 of the 200653 weight systems with $l = 5$ do not have the IP property, leading to the results given. $\square$

Remark: The 95 IP weight systems for $l = 4$ are precisely the well known 95 weight systems for weighted $\mathbb{P}^4$'s that have K3 hypersurfaces [44,45], whereas for $l = 5$ the 7555 weight systems corresponding to weighted $\mathbb{P}^4$'s that allow transverse polynomials [10,11] are just a small subset of the 184026 different IP weight systems.

Lemma 6: In dimensions $n = 1, 2, 3, 4$, the CWS with the IP property are the weight systems with $l = n + 1$ given in the previous lemma and, in addition, the following CWS:
$n = 2$: $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$
$n = 3$: The 21 CWS given in table 2
$n = 4$: 17320 CWS (cf. the second column of table 1)

Proof: By explicitly combining the structures of Lemma 3 with the IP weight systems of Lemma 5 and checking for the IP property of $\Delta(\mathbf{q})$. $\square$

2.3 The classification

As we saw in the previous subsections, every reflexive polyhedron must contain at least one minimal polytope corresponding to one of the CWS found there. Thus, by duality, every reflexive polyhedron must be a subpolyhedron of one of the $\Delta(\mathbf{q})$ on some suitable sublattice of the finest possible lattice M_{finest}. We start this section with analysing the question of which dual pairs of lattices can be chosen such that a dual pair of polyhedra is reflexive on them. Then we give various refinements of our original definition of minimality, and finally we present our results on the classification of reflexive polyhedra.

Given a dual pair of polytopes such that Δ has n_V vertices and n_F facets (a facet being a codimension 1 face), the dual polytope has n_V facets and n_F vertices.

Definition: The vertex pairing matrix (VPM) X is the $n_F \times n_V$ matrix whose entries are $X_{ij} = \langle \bar{V}_i, V_j \rangle$, where $\bar{V}_i$ and V_j are the vertices of Δ^* and Δ, respectively.

X_{ij} will be -1 whenever V_j lies on the i'th facet. Note that X is independent of the choice of a dual pair of bases in $N_\mathbb{R}$ and $M_\mathbb{R}$ but depends on the orderings of the vertices. If Δ is reflexive, then its VPM is obviously integer. In this case there are distinguished lattices M_{coarsest} and N_{coarsest}, generated by the vertices of Δ and Δ^*, respectively, and their duals N_{finest} and M_{finest}. Clearly any lattice M on which Δ is reflexive must fulfill $M_{\text{coarsest}} \subseteq M \subseteq M_{\text{finest}}$.

Lemma 7: If $\Delta \subset M_\mathbb{R} \simeq \mathbb{R}^n$ is a polytope with the IP property such that its VPM X is integer, the following statements hold:
X can be decomposed as $X = \tilde{W} \cdot \tilde{D} \cdot \tilde{U} = W \cdot D \cdot U$, where $\tilde{W}$ is a $GL(n_F, \mathbb{Z})$ matrix, $\tilde{U}$ is a $GL(n_V, \mathbb{Z})$ matrix and $\tilde{D}$ is an $n_F \times n_V$ matrix such that the first n diagonal elements are positive integers whereas all other elements are zero; W, D and U are the obvious $n_F \times n$, $n \times n$ and $n \times n_V$ submatrices.
The lattices $M \subset M_\mathbb{R}$ on which Δ is reflexive are in one to one correspondence with decompositions $D = T \cdot S$, where T and S are upper triangular integer matrices with positive diagonal elements and with $0 \le T_{ji} < T_{ii}$. Then Δ as a lattice polyhedron on M is isomorphic to the polytope in $\mathbb{Z}^n$ whose vertices are given by the columns of $S \cdot U$ and Δ^* is isomorphic to the polytope in $\mathbb{Z}^n$ whose vertices are given by the lines of $W \cdot T$. In particular, Δ on M_{finest} corresponds to $D \cdot U$, Δ on M_{coarsest} corresponds to U, Δ^* on N_{finest} corresponds to $W \cdot D$ and Δ^* on N_{coarsest} corresponds to W.

Proof: By recombining the lines and columns of X in the style of Gauss's algorithm for solving systems of linear equations, we can turn X into an $n_F \times n_V$ matrix $\tilde{D}$ with non-vanishing elements only along the diagonal. But recombining lines just corresponds to left multiplication with some $GL(n_F, \mathbb{Z})$ matrix, whereas recombining columns corresponds to right multiplication with some $GL(n_V, \mathbb{Z})$ matrix. Keeping track of the inverses of these matrices, we successively create decompositions $X = \tilde{W}^{(n)} \cdot \tilde{D}^{(n)} \cdot \tilde{U}^{(n)}$ (with $\tilde{W}^{(0)} = 1$, $\tilde{D}^{(0)} = X$ and $\tilde{U}^{(0)} = 1$). We denote the matrices resulting from the last step by $\tilde{W}$, $\tilde{D}$ and $\tilde{U}$. $\tilde{W}$ and $\tilde{U}$ being regular matrices and the rank of X being n, it is clear that $\tilde{D}$ has only n non-vanishing elements which can be taken to be the first n diagonal elements.

In the same way as we defined an embedding of $M_{\mathbb{R}}$ in $\mathbb{R}^k$ lemma 4, we now define an embedding in $\mathbb{R}^{n_F}$ such that M_{finest} is isomorphic to the sublattice of $\mathbb{Z}^{n_F}$ determined by the linear relations among the $\bar{V}_i$. In this context the X_{ij} are just the embedding coordinates of the V_j. The $n_F \times n_F$ matrix $\tilde{W}$ effects a change of coordinates in $\mathbb{Z}^{n_F}$ so that Δ now lies in the lattice spanned by the first d coordinates. Thus we can interpret the columns of $D \cdot U$ as the vertices of Δ on M_{finest}. Similarly, the lines of $W \cdot D$ are coordinates of the vertices of Δ^* on N_{finest}, whereas U and W are the corresponding coordinates on the coarsest possible lattices.

Denoting the generators of M_{coarsest} by $\vec{E}_i$ and the generators of M_{finest} by $\vec{e}_i$, we have $\vec{E}_i = \vec{e}_j D_{ji}$. An intermediate lattice will have generators $\vec{\mathcal{E}}_i = \vec{e}_j T_{ji}$ such that the $\vec{E}_i$ can be expressed in terms of the $\vec{\mathcal{E}}_j$, amounting to $\vec{E}_i = \vec{\mathcal{E}}_j S_{ji} = \vec{e}_k T_{kj} S_{ji}$ with some integer matrix S. This results in the condition $D_{ki} = T_{kj} S_{ji}$. In order to get rid of the redundancy coming from the fact that the intermediate lattices can be described by different sets of generators, one may proceed in the following way: $\vec{\mathcal{E}}_1$ may be chosen as a multiple of $\vec{e}_1$ (i.e., $\vec{\mathcal{E}}_1 = \vec{e}_1 T_{11}$). Then we choose $\vec{\mathcal{E}}_2$ as a vector in the $\vec{e}_1$-$\vec{e}_2$-plane (i.e., $\vec{\mathcal{E}}_1 = \vec{e}_1 T_{12} + \vec{e}_2 T_{22}$) subject to the condition that the lattice generated by $\vec{\mathcal{E}}_1$ and $\vec{\mathcal{E}}_2$ should be a sublattice of the one generated by $\vec{E}_1$ and $\vec{E}_2$, which is equivalent to the possibility of solving $T_{kj} S_{ji} = D_{ki}$ for integer matrix elements of S. We may avoid the ambiguity arising by the possibility of adding a multiple of $\mathcal{E}_1$ to $\mathcal{E}_2$ by demanding $0 \leq T_{12} < T_{11}$. We can choose the elements of T column by column (in rising order). For each particular column i we first pick T_{ii} such that it divides D_{ii}; then $S_{ii} = D_{ii}/T_{ii}$. Then we pick the T_{ji} with j decreasing from $i-1$ to 1. At each step the j'th line of $T \cdot S = D$,

$$T_{ji} S_{ii} + \sum_{j<k<i} T_{jk} S_{ki} + T_{jj} S_{ji} = 0, \tag{6}$$

must be solved for the unknown T_{ji} and S_{ji} with the extra condition $0 \leq T_{ji} < T_{ii}$ ensuring

that we get only one representative of each equivalence class of bases. □

At this point we have, in principle, all the ingredients that we need for a complete classification of reflexive polyhedra. We simply have to construct all subpolyhedra with integer VPM of all $\Delta(\mathbf{q})$ with $\mathbf{q}$ being one of our IP CWS, and apply lemma 7. Both for theoretical and for practical reasons, however, it is interesting to reduce the number of polyhedra used as a starting point in our scheme. To this end we will give various refinements of our original definition of minimality, preceded by a useful lemma on the structure of $\Delta(\mathbf{q})$.

Lemma 8: For $n \le 4$, $\Delta(\mathbf{q})$ is reflexive whenever it has the IP property.

Proof: This fact was proved in [26] and later explicitly confirmed by our computer programs. □

Definition: Let $\nabla \subset N_{\mathbb{R}}$ be a minimal lattice polyhedron such that Δ, the convex hull of $\nabla^* \cap M$, also has the IP property. Then we say that

∇ has the span property if the vertices of ∇ are also vertices of Δ^*.

∇ is lp-minimal: If we remove one of the vertices of ∇, the convex hull of the remaining set of *lattice points* of ∇ does not have the IP property.

∇ is very minimal: If we remove one of the vertices of ∇ from the set of lattice points of Δ^*, the convex hull of the remaining lattice points of Δ^* does not have the IP property.

A CWS $\mathbf{q}$ is said to have one of the above properties if the corresponding ∇ on N_{coarsest} has it.

A reflexive polytope $\Delta \subset M_{\mathbb{R}}$ is called r-maximal (and its dual $\Delta^* \subset N_{\mathbb{R}}$ r-minimal) if it is not contained in any other reflexive polytope.

A CWS $\mathbf{q}$ is called r-minimal if $\Delta(\mathbf{q})$ is r-maximal.

The name 'span property' refers to the fact that our definition is equivalent to the statement that the hyperplanes in $M_{\mathbb{R}}$ dual to the vertices of ∇ are spanned by points of Δ. The following lemma clarifies the relations between the various definitions of minimality and the ways in which these definitions can be used to refine our classification scheme. It also answers the question of how many CWS of the various minimality types exist.

Lemma 9:

a) For every reflexive polytope $\Delta \subset M_{\mathbb{R}}$, there exists at least one CWS $\mathbf{q}$ with the span property such that Δ is a subpolyhedron of the convex hull of $\nabla^* \cap M$ and M is a sublattice of M_{finest}.

b) For every reflexive polytope $\Delta \subset M_{\mathbb{R}}$, there exists at least one lp-minimal CWS $\mathbf{q}$ such that Δ is a subpolyhedron of the convex hull of $\nabla^* \cap M$ and M is a sublattice of M_{finest}.

c) If $\mathbf{q}$ is very minimal, $\Delta(\mathbf{q})$ is not a subpolyhedron of $\Delta(\mathbf{q}')$ for any $\mathbf{q}'$ corresponding to a minimal polytope different from the one defined by $\mathbf{q}$.

d) A very minimal polytope is lp-minimal and has the span property.

e) For every reflexive polytope $\Delta \subset M_\mathbb{R} \simeq \mathbb{R}^n$ with $n \leq 4$, there exists at least one r-minimal CWS $\mathbf{q}$ such that Δ is a subpolyhedron of the convex hull of $\nabla^* \cap M$ and M is a sublattice of M_{finest}.

f) For $n \leq 4$, a CWS $\mathbf{q}$ is r-minimal if and only if it is very minimal.

g) For $n \leq 3$ (but not for $n = 4$), every lp-minimal CWS has the span property.

h) The very minimal CWS for $n = 2$ are $\{(1,1,1)\}, \{(1,1,2)\}$ and $\{(1,1,0,0),(0,0,1,1)\}$. The remaining IP weight system $\{(1,2,3)\}$ has the span property but is not lp-minimal.

i) For $n = 3$ the minimality type is indicated in tables 2 and 3.

j) For $n = 4$ the numbers of CWS of the different minimality types are given in table 1.

Proof: a) By dropping vertices from Δ^* one can always arrive at a minimal polytope $\nabla \subseteq \Delta^*$ and the corresponding CWS.

b) By dropping lattice points from Δ^* one can always arrive at an lp-minimal (and therefore also minimal) polytope $\nabla \subseteq \Delta^*$ and the corresponding CWS.

c) If $\Delta(\mathbf{q})$ were a subpolyhedron of $\Delta(\mathbf{q}')$ for some $\mathbf{q}'$ other than $\mathbf{q}$, then $(\Delta(\mathbf{q}))^*$ would contain (but not be equal to) $(\Delta(\mathbf{q}'))^*$, which is impossible by the definition of $\mathbf{q}$ being very minimal.

d) By definition, $\Delta \subseteq \nabla^*$, implying $\nabla \subseteq \Delta^*$. Very minimal implies span: If a vertex of ∇ were not a vertex of Δ^*, it would be in the convex hull of the remaining lattice points of Δ^* which then would be equal to Δ^* and hence have the IP property, thus violating the assumption that ∇ is very minimal. The fact that very minimal implies lp-minimal is obvious from comparing the different definitions.

e) With a), we can find a CWS $\mathbf{q}^{(1)}$ such that Δ is a subpolyhedron of $\Delta(\mathbf{q}^{(1)})$, possibly on a sublattice. By lemma 8, $\Delta(\mathbf{q}^{(1)})$ is reflexive. If $\mathbf{q}^{(1)}$ is not r-minimal, $\Delta(\mathbf{q}^{(1)})$ is a proper subpolyhedron of some other reflexive polyhedron $\Delta^{(1)}$ for which we can find a CWS $\mathbf{q}^{(2)}$ as before. As the number of lattice points of $\Delta(\mathbf{q}^{(i)})$ increases in every step, this process has to terminate; thus $\mathbf{q}^{(i)}$ must be r-minimal for some i.

f) Because of c), every very minimal CWS is r-minimal. The fact that every r-minimal CWS is very minimal was checked explicitly by our computer programs.

g) – j) By explicit checks, for $n \geq 3$ with the help of our computer programs. $\square$

To end this section, we now give the results of the application of our classification scheme

for various dimensions.

Proposition 1: For $n = 2$ there are 16 reflexive polyhedra up to linear isomorphisms. All of them are subpolyhedra of $\Delta(\mathbf{q})$ where $\mathbf{q}$ is one of the three very minimal CWS.

Proof: The classification of 2-dimensional reflexive polyhedra has been established for a while (see, e.g., [46,47]) and is easily reproduced within our scheme. The second fact can be checked explicitly. □

Proposition 2: For $n = 3$ there are 4319 reflexive polyhedra up to linear isomorphisms. 4318 of them are subpolyhedra of $\Delta(\mathbf{q})$ where $\mathbf{q}$ is one of the very minimal CWS of tables 2 and 3. The remaining one is the convex hull of $\nabla^* \cap M$, where ∇ is determined by the weight system $(1, 1, 1, 1)$ and M is a $\mathbb{Z}_2$ sublattice of M_{finest}.

Proof: We explicitly constructed all subpolyhedra with integer VPMs of the $\Delta(\mathbf{q})$ coming from very minimal CWS with the help of a computer program and checked that polyhedra coming from CWS that are not very minimal are contained in the list of reflexive subpolyhedra of the $\Delta(\mathbf{q})$ for very minimal CWS. Application of lemma 7 produced the last polyhedron. □

Proposition 3: For $n = 4$ there are more than 473.8 million reflexive polyhedra up to linear isomorphisms. In addition to $\Delta(\mathbf{q})$ with $\mathbf{q}$ one of 308 the r-minimal CWS, there are at least 25 further r-maximal polyhedra.

Proof: Our computer programs have already produced more than 473.8 million different reflexive polyhedra. The 25 additional r-maximal polyhedra were obtained by applying lemma 7 to the original 308 r-maximal polytopes and checking for r-minimality of the duals on the various lattices allowed by lemma 7. □

3 Geometric interpretation of our classification results

We now want to discuss what our results on the classification of reflexive polyhedra imply for Calabi–Yau manifolds that are hypersurfaces in toric varieties.

The lattice points of a reflexive polyhedron Δ encode the monomials occurring in the description of the hypersurface in a variety V_Σ whose fan Σ is determined by a triangulation of the dual polyhedron Δ^*. For details of what a fan is and how it determines a toric variety,

it is best to look up a standard textbook [48, 49]. There is one particular approach to the description of toric varieties, however, which cannot be found there. This is the description in terms of homogeneous coordinates [50], which is the one most useful for applications in physics, and which also exhibits in the clearest way the significance of the weight systems that we used in the context of our classification scheme. We will briefly present this approach and show how Calabi–Yau manifolds are constructed in this setup and then we will proceed to explain some of the consequences of our results in terms of geometry.

Given a fan Σ in $N_{\mathbb{R}}$, it is possible to assign a global homogeneous coordinate system to V_Σ in a way similar to the usual construction of $\mathbb{P}^n$. To this end one assigns a coordinate z_k, $k = 1, \cdots, K$ to each one dimensional cone in Σ. If the primitive generators $v_1, \cdots, v_K$ of these one dimensional cones span $N_{\mathbb{R}}$, then there must be $K - n$ independent linear relations of the type $\sum_k w_j^k v_k = 0$. These linear relations are used to define equivalence relations of the type

$$(z_1, \cdots, z_K) \sim (\lambda^{w_j^1} z_1, \cdots, \lambda^{w_j^K} z_K), \qquad j = 1, \cdots, K - n \tag{7}$$

on the space $\mathbb{C}^K \setminus Z_\Sigma$. The set Z_Σ is determined by the fan Σ in the following way: It is the union of spaces $\{(z_1, \cdots, z_K) : \ z_i = 0 \ \forall i \in I\}$, where the index sets I are those sets for which $\{v_i : \ i \in I\}$ does not belong to a cone in Σ. Thus $(\mathbb{C}^*)^K \subset \mathbb{C}^K \setminus Z_\Sigma \subset \mathbb{C}^K \setminus \{0\}$. Then $V_\Sigma = (\mathbb{C}^K \setminus Z_\Sigma)/((\mathbb{C}^*)^{(K-n)} \times G)$, where the $K - n$ copies of $\mathbb{C}^*$ act by the equivalence relations given above and the finite abelian group G is the quotient of the N lattice by the lattice generated by the v_k. We will usually consider the case where G is trivial. In this approach the toric divisors D_k are determined by the equations $z_k = 0$.

The construction of a Calabi–Yau hypersurface from a reflexive polyhedron proceeds in the following way: We take Δ to be a reflexive polyhedron in $M_{\mathbb{R}}$, $\Delta^* \subset N_{\mathbb{R}}$ its dual, and Σ a fan defined by a maximal triangulation of Δ^*. This means that the integer generators $v_1, \cdots, v_K$ of the one dimensional cones are just the integer points (except the origin) of Δ^*. The polynomial whose vanishing determines the Calabi–Yau hypersurface takes the form

$$\sum_{x \in \Delta \cap M} a_x \prod_{k=1}^{K} z_k^{\langle v_k, x \rangle + 1}. \tag{8}$$

It is easily checked that it is quasihomogeneous with respect to all $K - n$ relations of (7) with degrees $d_j = \sum_{k=1}^K w_j^k$, $j = 1, \cdots K - n$. Note how the reflexivity of the polyhedron ensures that the exponents are nonnegative.

By [15], the Hodge numbers h_{11} and $h_{1,n-2}$ are known, and in [35] the remaining Hodge numbers of the type h_{1i} were calculated. For a hypersurface of dimension $n-1 \geq 2$ these formulas can be summarised as

$$\begin{aligned} h_{1i} &= \delta_{1i}\left(l(\Delta^*) - n - 1 - \sum_{\mathrm{codim}\theta^*=1} l^*(\theta^*)\right) + \delta_{n-2,i}\left(l(\Delta) - n - 1 - \sum_{\mathrm{codim}\theta=1} l^*(\theta)\right) \\ &\quad + \sum_{\mathrm{codim}\theta^*=i+1} l^*(\theta^*) l^*(\theta) \end{aligned} \tag{9}$$

for $1 \leq i \leq n-2$, where l denotes the number of integer points of a polyhedron and l^* denotes the number of interior integer points of a face. These formulas are invariant under the simultaneous exchange of Δ with Δ^* and h_{1i} with $h_{1,n-i}$ so that Batyrev's construction is manifestly mirror symmetric (at least at the level of Hodge numbers). For $n \leq 4$, the generic $(n-1)$-dimensional Calabi-Yau hypersurface in the family defined by Δ will be smooth [15] and the meaning of these numbers is unambiguous. For $n \geq 5$, the Calabi-Yau variety may have singularities that do not allow a crepant blow-up. In this case we refer the reader to refs. [35] for a discussion of the precise meaning of the Hodge numbers resulting from eq. (9).

In the case of a K3 surface there is only one such number, namely h_{11}, which is well known always to be equal to 20. Contrary to the case of higher dimensional Calabi-Yau manifolds, this number is not the same as the Picard number, which is given by [15]

$$\mathrm{Pic} = l(\Delta^*) - 4 - \sum_{\text{facets } \theta^* \text{ of } \Delta^*} l^*(\theta^*) + \sum_{\text{edges } \theta^* \text{ of } \Delta^*} l^*(\theta^*) l^*(\theta). \tag{10}$$

Mirror symmetry for K3 surfaces is usually interpreted in terms of families of lattice polarized K3 surfaces (see, e.g., [51] or [52]). In this context the Picard number of a generic element of a family and the Picard number of a generic element of the mirror family add up to 20. The fact that the Picard numbers for toric mirror families add up to $20 + \sum l^*(\theta^*) l^*(\theta)$ indicates that our toric models occupy rather special loci in the total moduli spaces.

If a polyhedron Δ_1 contains a polyhedron Δ_2, then the definition of duality implies $\Delta_1^* \subset \Delta_2^*$. Therefore the variety determined by the fan over Δ_1^* may be obtained from the variety determined by the fan over Δ_2^* by blowing down one or several divisors. If we perform this blow-down while keeping the same monomials (those determined by Δ_2), we obtain a generically singular hypersurface. This hypersurface can be desingularised by varying the complex structure in such a way that we now allow monomials determined by Δ_1. Thus the classes of Calabi-Yau hypersurfaces determined by polyhedra Δ_1 and Δ_2, respectively, can be said to be connected

whenever Δ_1 contains Δ_2 or vice versa. More generally, if there is a chain of polyhedra Δ_i such that Δ_i and Δ_{i+1} are connected in the sense defined above, we call the hypersurfaces corresponding to any two elements of the chain connected.

We can easily check for connectedness as a by-product of our classification scheme: For each new CWS $\mathbf{q}$ we check explicitly that at least one of the subpolyhedra of $\Delta(\mathbf{q})$ has been found before. Connectedness of the corresponding list of 4318 polytopes in three dimensions follows from the fact that this is always the case. Connectedness of all 3-d reflexive polyhedra follows from the fact that the last polytope that we only found on a sublattice contains 679 reflexive proper subpolytopes that were found before. In the same way all of the four dimensional polyhedra that we have found so far form a connected web.

As we saw in the previous section, every three dimensional reflexive polytope $\Delta^* \subset N_{\mathbb{R}}$ contains one of 16 r-minimal polyhedra as a subpolytope on the same lattice. Therefore, the fan of any toric ambient variety determined by a maximal triangulation of a reflexive polyhedron is a refinement of one of the corresponding 16 fans. In other words, any such toric ambient variety is given by the blow-up of one of the following 16 spaces (cf. tables 2 and 3 and proposition 2):

– $\mathbb{P}^3$,
– $\mathbb{P}^3/\mathbb{Z}_2$,
– 8 different weighted projective spaces $\mathbb{P}^2_{(q_1,q_2,q_3)}$,
– $\mathbb{P}^2 \times \mathbb{P}^1$,
– $\mathbb{P}^2_{(1,1,2)} \times \mathbb{P}^1$,
– 3 further double weighted spaces, and
– $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Each of the three spaces with ‘overlapping weights’ allows two distinct bundle structures: The first one can be interpreted as a $\mathbb{P}^2$ bundle in two distinct ways, the second one as a $\mathbb{P}^2$ bundle or a $\mathbb{P}^2_{(1,1,2)}$ bundle, and the third one can be interpreted as a $\mathbb{P}^2_{(1,1,2)}$ bundle in two distinct ways. In each case the base space is $\mathbb{P}^1$.

Let us end this section with briefly discussing a few of the most interesting objects in our lists. There are precisely two mirror pairs with Picard numbers 1 and 19, respectively. One of them is the quartic hypersurface in $\mathbb{P}^3$ with Picard number 1, together with its mirror of Picard number 19, which is also the model whose Newton polytope is the only reflexive polytope with only 5 lattice points. This model corresponds to a blow-up of a $\mathbb{Z}^4 \times \mathbb{Z}^4$ orbifold of $\mathbb{P}^3$. The

blow-up of six fixed lines $z_i = z_j$ by three divisors each yields 18 exceptional divisors leading to the total Picard number of 19. The other mirror pair with Picard numbers 1 and 19 consists of the hypersurface in $\mathbb{P}^3_{(1,1,1,3)}$ of degree 6 and an orbifold of the same model, with Newton polyhedra with 39 and 6 points, respectively. This polyhedron is also one of the two 'largest' polyhedra in the sense that there is no reflexive polytope in three dimensions with more than 39 lattice points. The other polyhedron with the maximal number of 39 points is the Newton polytope of the hypersurface of degree 12 in $\mathbb{P}^3_{(1,1,4,6)}$. This model leads to the description of elliptically fibered K3 surfaces that is commonly used in F-theory applications [22, 29, 30], with the elliptic fiber embedded in a $\mathbb{P}^2_{(1,2,3)}$ by a Weierstrass equation. The mirror family of this class of models can be obtained by forcing two E_8 singularities into the Weierstrass model and blowing them up. The resulting hypersurface allows also a different fibration structure which can develop an $SO(32)$ singularity; thereby this model is able to describe the F-theory duals of both the $E_8 \times E_8$ and the $SO(32)$ heterotic strings with unbroken gauge groups in 8 dimensions [53].

In four dimensions there is a unique 'largest' object, determined by the weight system $(1, 1, 12, 28, 42)/84$. It has the maximum number, namely 680, of lattice points and the corresponding Calabi–Yau threefold has the Hodge numbers $h_{11} = 11$ and $h_{12} = 491$. The latter is the largest single Hodge number in our list, and the value of $|\chi| = |2(h_{11} - h_{12})| = 960$ is also maximal, the only other object with the same values being the mirror. F-theory compactifications of the latter lead to the theories with the largest known gauge groups in six dimensions [53].

Another interesting object that we encountered is the 24-cell, a self dual polytope with 24 vertices, which leads to a self mirror Calabi–Yau manifold with Hodge numbers (20,20). It has the maximal symmetry order $1152 = 24 * 48$ among all 4 dimensional reflexive polytopes and arises as a subpolytope of the hypercube. It is a Platonic solid that contains the Archimedian cuboctahedron (with symmetry order 48) as a reflexive section through the origin parallel to one of its 24 bounding octahedra. Note that in our context symmetries are realized as lattice isomorphisms, i.e. as subgroups of $GL(n, \mathbb{Z})$, and not as rotations.

The polytope with the largest order, namely 128, of $M_{\text{finest}}/M_{\text{coarsest}}$ is determined by the weight system $(1, 1, 1, 1, 4)/8$. For the Newton polytope of the quintic hypersurface in $\mathbb{P}^4$, this order is 125. There is a well known $\mathbb{Z}_5$ orbifold of the quintic with Hodge numbers (1,21) which is quite peculiar from the lattice point of view: Although the N lattice is not the lattice N_{finest}

generated by the vertices of Δ^*, the only lattice points of Δ^* are its vertices and the IP. Thus it provides an example where the N lattice is not even generated by the lattice points of Δ^*. This can only happen in more than 3 dimensions: As a lattice triangle with 3 lattice points is always regular (i.e. it has the minimal volume 1 in lattice units) and there are no lattice hyperplanes between a facet and the IP because of reflexivity, the vertices of any triangle of a maximal triangulation of a 2-dimensional facet of a 3-dimensional polytope provide a lattice basis.

4 Fibrations

In this section we want to discuss fibrations of hypersurfaces of holonomy $SU(n-1)$ in n-dimensional toric varieties where the generic fiber is an $(n_\mathrm{f}-1)$–dimensional variety of holonomy $SU(n_\mathrm{f}-1)$. In other words, it will apply to elliptic fibrations of K3 surfaces, CY threefolds, CY fourfolds, etc., to K3 fibrations of CY k-folds with $k \geq 3$, to threefold fibrations of fourfolds, and so on. The main message is that the structures occurring in the fibration are reflected in structures in the N lattice: The fiber, being an algebraic subvariety of the whole space, is encoded by a polyhedron Δ^*_f which is a subpolyhedron of Δ^*, whereas the base, which is a projection of the fibration along the fiber, can be seen by projecting the N lattice along the linear space spanned by Δ^*_f. We will first give a general discussion and then explain how descriptions in terms of CWS may be useful for identifying and/or encoding fibration structures.

4.1 Fibrations and reflexive polyhedra

Assume that Δ^* contains a lower-dimensional reflexive subpolyhedron $\Delta^*_\mathrm{f} = (N_\mathrm{f})_\mathbb{R} \cap \Delta^*$ with the same interior point. This allows us to define a dual pair of exact sequences

$$0 \to N_\mathrm{f} \to N \to N_\mathrm{b} \to 0 \tag{11}$$

and

$$0 \to M_\mathrm{b} \to M \to M_\mathrm{f} \to 0, \tag{12}$$

and corresponding sequences for the underlying real vector spaces. We can convince ourselves that the image of Δ under $M_\mathbb{R} \to (M_\mathrm{f})_\mathbb{R}$ is dual to Δ^*_f in the following way: We choose a basis

e^j, $j = 1, \dots n$ of N such that N_f is generated by the e^j with $1 \le j \le n_\mathrm{f}$ and define e_i to be the dual basis. Then

$$\Delta_\mathrm{f} = \{(x^1, \cdots, x^{n_\mathrm{f}}) : \exists\, x^{n_\mathrm{f}+1}, \dots, x^n \text{ with } (x^1, \cdots, x^n) \in \Delta)\}, \tag{13}$$

$$(\Delta^*)_\mathrm{f} = \{(y_1, \cdots, y_{n_\mathrm{f}}) : (y_1, \cdots, y_{n_\mathrm{f}}, 0) \in \Delta^*\}, \tag{14}$$

and the duality of these two polytopes is easily checked.

Let us also assume that the image Σ_b of Σ under $\pi : N \to N_\mathrm{b}$ defines a fan in N_b. This is certainly not true for arbitrary triangulations of Δ^*. Constructing fibrations, one should rather build a fan Σ_b from the images of the one-dimensional cones in Σ and try to construct a triangulation of Σ and thereby of Δ^* that is compatible with the projection. It would be interesting to know whether this is always possible whenever the intersection of a reflexive polyhedron with a linear subspace of $N_\mathbb{R}$ is again reflexive.

The set of one-dimensional cones in Σ_b is the set of images of one-dimensional cones in Σ that do not lie in N_f. The image of a primitive generator v_i of a cone in Σ is the origin or a positive integer multiple of a primitive generator $\tilde v_j$ of a one-dimensional cone in Σ_b. Thus we can define a matrix r^i_j, most of whose elements are 0, through $\pi v_i = r_i^j \tilde v_j$ with $r_i^j \in \mathbb{N}$ if πv_i lies in the one-dimensional cone defined by $\tilde v_j$ and $r_i^j = 0$ otherwise. Our base space is the multiply weighted space determined by

$$(\tilde z_1, \cdots, \tilde z_{\tilde K}) \sim (\lambda^{\tilde w_j^1} \tilde z_1, \cdots, \lambda^{\tilde w_j^{\tilde K}} \tilde z_{\tilde K}), \qquad j = 1, \cdots, \tilde K - \tilde n, \tag{15}$$

where $\tilde n = n - n_\mathrm{f}$ and the $\tilde w_j^i$ are any integers such that $\sum_i \tilde w_j^i \tilde v_i = 0$. The projection map from V_Σ (and, as we will see, from the Calabi–Yau hypersurface) to the base is given by

$$\tilde z_i = \prod_j z_j^{r_j^i}. \tag{16}$$

This is well defined: $z_j \to \lambda^{w_k^j} z_j$ leads to $\tilde z_i \to \lambda^{w_k^j r_j^i} \tilde z_i$ which is among the good equivalence relations because applying π to $\sum w_k^j v_j = 0$ gives $\sum w_k^j r_j^i \tilde v_i = 0$.

A generic point in the base space will have $\tilde z_i \neq 0$ for all i, implying $z_i \neq 0$ for all $v_i \notin \Delta_\mathrm{f}^*$. The choice of a specific point in V_{Σ_b} and the use of all equivalence relations except for those involving only $v_i \in \Delta_\mathrm{f}^*$ allows to fix all z_i except for those corresponding to $v_i \in \Delta_\mathrm{f}^*$. Thus the preimage of a generic point in V_{Σ_b} is indeed a variety in the moduli space determined by Δ_f^*.

What we have seen so far is just that V_Σ is a fibration over V_{Σ_b} with generic fiber V_{Σ_f} (this is actually the statement of an exercise on p. 41 of ref. [48]) and how this fibration structure manifests itself in terms of homogeneous coordinates. Now we also want to see how this can be extended to hypersurfaces. To this end note that if $v_k \in \Delta^*_\mathrm{f}$ then $\langle v_k, x\rangle$ only depends on the equivalence class $[x] \in M_\mathrm{f}$ of x under

$$x \sim y \quad \text{if} \quad x - y \in M_\mathrm{b}. \tag{17}$$

Thus we may rewrite eq. (8) as

$$p = \sum_{[x]\in\Delta_\mathrm{f}\cap M_\mathrm{f}} a'_{[x]} \prod_{v_k\in\Delta^*_\mathrm{f}} z_k^{\langle v_k,[x]\rangle+1} \qquad \text{with} \qquad a'_{[x]} = \sum_{x\in[x]} a_x \prod_{v_k\notin\Delta^*_\mathrm{f}} z_k^{\langle v_k,x\rangle+1}. \tag{18}$$

In each coordinate patch for V_{Σ_b} this is just an equation for the fiber with coefficients that are polynomial functions of coordinates of the base space.

Whenever a one-dimensional cone (with primitive generator $\tilde{v}_i$) in Σ_b is the image of more than one one-dimensional cone in Σ, the fiber becomes reducible over the divisor $\tilde{z}_i = 0$ determined by v_i. Different components of the fiber correspond to different equations $z_j = 0$ with $\pi v_j = r^i_j \tilde{v}_i$. The intersection patterns of the different components of the reducible fibers are crucial for understanding enhanced gauge symmetries in type IIA string theory [52, 54] and F-theory [22, 29, 30]. Blowing down the corresponding subvarieties (and hence making the Calabi–Yau space itself singular) leads to the appearance of non-perturbative enhanced gauge groups whose Lie algebras are determined by the intersection patterns of the components of the fibers. In terms of the N lattice, the occurrence of enhanced gauge groups can be easily inferred by studying the preimage of a one-dimensional cone in Σ_b. In particular, as noted by Candelas and Font [55], under favorable circumstances the Dynkin diagram of the corresponding Lie algebra can be seen directly in the toric diagram in the N lattice.

4.2 Fibrations and weight systems

As for polyhedra, weight systems provide a very useful and economic tool for constructing and describing fibrations. We only consider toric CY fibrations which require a reflexive section through the origin of $\Delta^* \subset N_\mathbb{R}$ whose dimension is equal to n_f for $(n_\mathrm{f}-1)$-dimensional CY-fibers. In the M lattice this corresponds to a projection onto the dual reflexive polyhedron along an $(n - n_\mathrm{f})$-dimensional subspace. Hence, on either side, we need to specify a linear

subspace $(N_\mathrm{f})_\mathbb{R} \subset N_\mathbb{R}$ or $(M_\mathrm{b})_\mathbb{R} \subset M_\mathbb{R}$. This can be done, for example, by singling out vectors that span the subspace or by representing the subspace as an intersection of hyperplanes.

If the polyhedron is given in terms of some CWS it is natural to try to specify this linear subspace by using some part of the weight information. Note that a lower index i in a CWS $n_i^{(j)}$ corresponds to a vertex of the minimal polytope ∇, or, by duality, to a bounding hyperplane in the M lattice (which, if Δ is embedded into $\mathbb{R}^k$, is the intersection of the coordinate hyperplane $a_i = 0$, or $x_i = -1$, with the affine or linear subspace that supports Δ). An upper index j corresponds to a simplex in the CWS and, hence, to the linear subspace spanned by that simplex. Actually this can be regarded as a special case of the former correspondence, since the subspace that is spanned by a simplex S^j is generated by those vectors of ∇ for which $n_i^{(j)} \neq 0$.

The simplest case is therefore the situation where a subset of the IP simplex structure of the minimal polytope ∇ provides a weight system for Δ_f. In turn, we can engineer fibrations where the fiber corresponds to a certain weight system if we start by generating combined weight systems $\mathbf{q}$ with a given subsystem $\mathbf{q}_\mathrm{f}$. As usual, one has to check that $\Delta(\mathbf{q})$ is reflexive, which in up to 4 dimensions is equivalent to the IP property (in the case of Calabi–Yau 4-fold CWS with $\Delta(\mathbf{q})$ IP but not reflexive we can proceed with reflexive subpolytopes $\Delta' \subset \Delta(\mathbf{q})$). Because of the additional equations that come from the extended CWS $\mathbf{q}$, the set of solutions to eq. (1) is smaller than those for $\mathbf{q}_\mathrm{f}$ when we disregard the additional coordinates. Hence the intersection of Δ^* with the subspace spanned by the vertices of ∇_f contains $\Delta(\mathbf{q}_\mathrm{f})^*$, but may be larger. We therefore obtain a fibration if the resulting n_f-dimensional polytope is reflexive.[1]

In the case of elliptic fibrations this is always true: We want to show that $\Delta_\mathrm{f}^* = \Delta^* \cap (N_\mathrm{f})_\mathbb{R}$ is reflexive. As we saw in section 4.1, Δ_f can be identified with the image of Δ under $M_\mathbb{R} \to (M_\mathrm{f})_\mathbb{R}$. As the image of $\mathbf{0}_M$ under $M \to M_\mathrm{f}$ is an IP of Δ_f, Δ_f has at least one IP. Because of the well known fact that a two dimensional lattice polytope is reflexive if it has precisely one IP, all that is left to show is that Δ_f cannot have more than one IP. As $\Delta_\mathrm{f}^* \supseteq \nabla_\mathrm{f}$ implies $\Delta_\mathrm{f} \subseteq \nabla_\mathrm{f}^*$ and ∇_f as a lattice polytope with a single IP is reflexive, ∇_f^* and hence Δ_f has precisely one IP, i.e. Δ_f is indeed reflexive. To obtain elliptic fibrations in Weierstrass form, as they are mostly used in F theory compactifications [22, 29, 30], we thus only need to take the weight system $\mathbf{q}_\mathrm{f} = (1, 2, 3)/6$ as the fiber part of a combined weight system and check for reflexivity of $\Delta(\mathbf{q})$.

[1] If this is not the case we could proceed by dropping vertices of Δ and trying to find a larger reflexive section, but this soon becomes ugly and in view of the abundance of weight systems it is hardly worth the effort.

It is no surprise that the situation we described does not cover the general case: Given an IP simplex structure of a minimal polytope $\nabla_{\mathrm{f}} \subseteq \Delta_{\mathrm{f}}^*$ it is not always possible to extend it to a simplex structure for a minimal polytope $\nabla \subseteq \Delta^*$. As a simple example in 4 dimensions we consider the points $V_1, \ldots, V_6 \in N$ with coordinates given by the columns of the matrix

$$(V_1, \ldots, V_6) = \begin{pmatrix} -1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{19}$$

The first three points provide a weight system (2,1,1) for an elliptic fiber such that Δ_{f}^* is supported by the 1–2 plane. V_1 is contained in the convex hull of $V_2, \ldots, V_6$, which is a minimal polytope ∇ as defined in section 2.1. The weight system for ∇ is $\mathbf{q} = (2, 2, 2, 1, 1)/8$ and we cannot use $(2, 1, 1)/4$ as part of a CWS corresponding to a minimal polyhedron ∇ with an IP simplex structure in the sense of section 2. In this situation it makes sense to use generalized IP simplex structures where the vertices of the IP simplices are lattice points (but not necessarily vertices) of Δ^* and we do not insist on the non-redundancy implied by our original definition of an IP simplex structure.

Having made this point we may use the linear relations among the vertices of the simplices (V_1, V_2, V_3) and $(V_2, V_3, V_4, V_5, V_6)$ to arrive at the CWS $\mathbf{q}^{(1)} = (2, 1, 1, 0, 0, 0)/4$ and $\mathbf{q}^{(2)} = (0, 2, 2, 2, 1, 1)/8$. A CWS of this type was not considered in our classification scheme because $V_1 = (V_2 + V_3)/2$ is redundant when combined with the vertices that correspond to $\mathbf{q}^{(2)}$. It does, however, lead to a perfectly sensible system of equations (1), the convex hull of whose solutions is ∇^* (in our example all polytopes are simplices). Actually, for $\mathbf{q} = (2, 2, 2, 1, 1)/8$ we find that $(\Delta(\mathbf{q}))^*$ has the seven lattice points $V_2, \ldots, V_6$, $\mathbf{0}$ and $(-1, 0, -1, 0)^T$, but no reflexive subpolytope. The CWS $\{\mathbf{q}^{(1)}, \mathbf{q}^{(2)}\}$, on the other hand, leads to a polytope $(\Delta(\mathbf{q}^1, \mathbf{q}^2))^*$ with nine lattice points and the reflexive subpolytope that we started with: The addition of V_1 refines the lattice generated by the vertices of ∇ in such a way that the convex hull of $V_2, \ldots, V_6$ on the finer N lattice now contains the additional lattice points V_1 and $-V_1$. As an aside we thus observe that a CWS corresponding to a generalized IP simplex structure may also be used to encode certain sublattices of M_{finest}. Probably most polytopes can be directly specified by using a generalized IP simplex structure and the corresponding CWS. A counterexample is given by the $\mathbb{Z}_5$ quotient of the quintic at the end of section 3, where the N lattice is not generated by $\Delta^* \cap N$. But in practice such a representation is only useful if the number of equations is small. In any case combined weight systems provide a simple construction for toric fibrations and can

always be used to specify reflexive sections.

4.3 Toric fibrations and weighted projective spaces

There is a more subtle way in which a fibration structure can be encoded in a weight system. It only works for codimension 1 fibers, but it is quite interesting for historical and practical reasons. When string dualities led to interest in K3 fibrations, the first examples were constructed in the context of weighted projective spaces [56, 57]. It turned out that these examples are indeed special cases of toric fibrations in the sense that they correspond to reflexive projections of Newton polyhedra of transversal hypersurfaces in weighted $\mathbb{P}^3$ or $\mathbb{P}^4$.

Actually, reflexive objects of codimension 1 were first observed on these Newton polyhedra, either as reflexive facets or as reflexive sections through the IP in the M lattice [55]. Since what we really need for a toric fibration is a reflexive section in $N_\mathbb{R}$, the question arises whether there is a reflexive projection of Δ onto one of its facets. A simple necessary condition for this is provided by the following observations. We work with the embedding space of lemma 4 and do not distinguish between objects in $M_\mathbb{R}$ and their images under the embedding map.

Lemma 10:

a) For a polytope Δ defined by a weight system n_i only facets that are supported by a lattice hyperplane $x_l = -1$ can have interior points.

b) If $\mathbf{y} = (y_1, \ldots, y_k) \in M$ has $y_l = -1$, $y_i \geq 0$ for $i \neq l$, the map $\pi_\mathbf{y} : M_\mathbb{R} \to M_\mathbb{R}$, $\pi_\mathbf{y}\mathbf{x} = \mathbf{x} + (x_l + 1)\mathbf{y}$ has the following properties:

It is a projection to the affine subspace $x_l = -1$, i.e. $\pi_\mathbf{y}^2 = \pi_\mathbf{y}$ and $\pi_\mathbf{y} M_\mathbb{R} = M_\mathbb{R} \cap \{x_l = -1\}$.

It respects the lattice structure, i.e. if $\mathbf{x} \in M$ then $\pi_\mathbf{y}\mathbf{x} \in M \cap \{x_l = -1\}$,

The image of Δ is the corresponding facet of Δ, i.e. $\pi_\mathbf{y}\Delta = \Delta \cap \{x_l = -1\}$.

c) There is a one-to-one correspondence between maps π with the same properties as in b) such that $\mathbf{0}$ gets mapped to an IP of the facet with $x_l = -1$ and partitions of the weight n_l by the remaining weights, i.e. $n_l = \sum_{i \neq l} y_i n_i$ where the y_i are nonnegative integers.

Proof: a) If an interior point of a facet is not on some hyperplane $x_l = -1$ all x_i must be nonnegative, but this is only possible for the interior point of Δ.

b) The first two statements follow directly from the definition of $\pi_\mathbf{y}$. $\pi_\mathbf{y}\Delta \supseteq \Delta \cap \{x_l = -1\}$ follows from the fact that $\pi_\mathbf{y}$ is a projection. For $\pi_\mathbf{y}\Delta \subseteq \Delta \cap \{x_l = -1\}$ we note that $\Delta \cap \{x_l = -1\}$ is the convex hull of the lattice points in $\{x_l = -1\}$ with $x_i \geq -1$ and that every vertex

of Δ gets mapped to such a lattice point.
c) If π is such a map, then we choose $\mathbf{y}$ to be the IP of the facet to which $\mathbf{0}$ is mapped (this implies $y_i \geq 0$ for $i \neq l$), and the partition of n_l follows from the fact that $\sum y_i n_i = 0$. Conversely, if $\mathbf{y}$ is defined by such a partition we have to show that it is interior to the facet. This follows from the facts that $\mathbf{0}$ is interior to Δ and $\mathbf{y} = \pi_{\mathbf{y}}\mathbf{0}$. □

A necessary condition for the existence of a reflexive projection of Δ onto one of its facets is therefore that one of the weights n_i has a *unique* partition in terms of the other weights. Using this criterion we found all such projections for single weight systems with $k \leq 5$ by first searching for weights with unique partitions and then checking reflexivity of the corresponding facets. The results are given in table 3 for the case of elliptic K3 surfaces and they are available on our web page [33] for K3-fibered Calabi–Yau manifolds (cf. table 4).

We can find a set of generators for $(N_{\mathrm{f}})_{\mathbb{R}}$ by solving the equation $\langle V, \mathbf{y}\rangle = 0$ for a general linear combination $V = \sum c_j V_j$ of the vertices V_j of ∇. Since $\langle V_i, \mathbf{y}\rangle = y_i$ we obtain the solutions $V_i' = V_i + y_i V_l$ for $i \neq l$. The linear relations among the V_i' are given by the corresponding subset of the original weights. In general they do not provide a weight system for the fiber because the points V_i' need not belong to Δ^*.

This is easy to see for the class of weights $(1, 1, 2n_3, 2n_4, 2n_5)$ that was considered by Klemm, Lerche and Mayr [56]. Here $V_2' = V_1 + V_2$ and $\langle V_2', \mathbf{x}\rangle = x_1 + x_2$ for $\mathbf{x} \in M$ with coordinates x_i. But $\sum x_i n_i = 0$ implies that $x_1 + x_2$ is even, so V_2' is not a primitive lattice vector in N and can be divided by 2, which leads to the weight system $(1, n_3, n_4, n_5)$ for the K3 fiber. The slightly more complicated example $(8, 4, 3, 27, 42)/84$ was given by Hosono, Lian and Yau [57]. The first weight has a unique partition with $y_2 = 2$ and $y_3 = y_4 = y_5 = 0$, so that $(N_{\mathrm{f}})_{\mathbb{R}}$ is spanned by $V_2' = V_2 + 2V_1$ and V_i with $i > 2$. This time $8x_1 + 4x_2 + 3x_3 + 27x_4 + 42x_5 = 0$ implies that $x_2 + 2x_1$ is a multiple of 3 and the primitive lattice vector $V_2'/3 \in \Delta^*$ leads to the weight system $(4, 1, 9, 14)$ for the fiber, which agrees with the normalized weights for the fiber given in [57]. If more of the coefficients y_i are nonvanishing, it is, of course, still possible to compute a weight system for the fiber, but this gets more tedious and we would also lose the direct connection with the original weights or we would have to introduce many redundant coordinates in a CWS.

Another strategy for identifying reflexive projections of $\Delta(q)$ that can be used in the codimension 1 case follows from the fact that such a projection either must be along a line parallel

to a facet or onto that facet whenever a facet has an interior point. If the number of facets with interior points is large enough this allows us to find all reflexive projections. The result of this analysis is indicated in the next-to-last column of tables 3 and 4. The K3 surfaces in table 2 are all elliptic, since their combinded weight systems contain 2 dimensional subsystems.

With our strategies to identify reflexive projections (onto facets) we generalized the results of [56, 57] and extended the scope from the transversal case to the complete list of 184026 IP weight systems, where we could identify 124701 fibrations. The efficiency of our approach can be inferred from the fact that we found 5370 fibrations for the 7555 transversal cases, wheras only 628 fibrations yielded to the methods of [57].

Acknowledgements: M.K. is partly supported by the Austrian Research Funds FWF grant Nr. P11582-PHY. The research of H.S. is supported by the European Union TMR project ERBFMRX-CT-96-0045.

Appendix: Various tables

IP simplex structure	total	span	lp-min.	r-min.
$S_1 = V_1V_2V_3V_4V_5$	184026	38730	16437	206
$S_1 = V_1V_2V_3V_4,\ S_2 = V_1V_2V_3'V_4'$	16040	6365	143	51
$S_1 = V_1V_2V_3V_4,\ S_2 = V_1V_2'V_3'$	1122	727	40	29
$S_1 = V_1V_2V_3,\ S_2 = V_1'V_2'V_3'$	6	6	3	3
$S_1 = V_1V_2V_3,\ S_2 = V_1V_2'V_3',\ S_3 = V_1V_2''V_3''$	36	36	4	4
$S_1, \ldots, S_{m-1}$ as for $n = 3$, $S_m = V_1^{(m)}V_2^{(m)}$	116	79	19	15
total	201346	45943	16646	308

Table 1: IP simplex structures and numbers of corresponding IP CWS for $n = 4$ (for $n = 3$, see lemma 3).

d	n_1	n_2	n_3	n_4	n_5		P	V	$\overline{P}$	$\overline{V}$
3	1	1	1	0	0					
3	1	0	0	1	1	r	30	5	6	5
3	1	1	1	0	0					
4	2	0	0	1	1	r	31	6	7	5
3	1	1	1	0	0					
4	1	0	0	2	1	s	23	7	8	6
3	1	1	1	0	0					
6	3	0	0	2	1	s	24	6	9	5
3	1	1	1	0	0					
6	2	0	0	3	1	s	21	5	9	5
3	1	1	1	0	0					
6	1	0	0	3	2	s	14	7	11	6
4	2	1	1	0	0					
4	2	0	0	1	1	r	35	5	7	5
4	2	1	1	0	0					
4	1	0	0	2	1	s	23	6	9	5
4	2	1	1	0	0					
6	3	0	0	2	1	s	27	5	9	5
4	1	2	1	0	0					
4	1	0	0	2	1	s	19	5	9	5
4	1	2	1	0	0					
6	3	0	0	2	1	s	18	6	12	5

d	n_1	n_2	n_3	n_4	n_5	n_6		P	V	$\overline{P}$	$\overline{V}$
4	1	2	1	0	0						
6	2	0	0	3	1		s	16	6	14	6
4	1	2	1	0	0						
6	1	0	0	3	2		s	12	6	14	6
6	3	2	1	0	0						
6	3	0	0	2	1		s	21	5	12	5
6	2	3	1	0	0						
6	2	0	0	3	1		s	15	5	15	5
6	2	3	1	0	0						
6	1	0	0	3	2		s	10	6	20	6
6	1	3	2	0	0						
6	1	0	0	3	2		s	9	5	18	5
3	1	1	1	0	0						
2	0	0	0	1	1		r	30	6	6	5
4	2	1	1	0	0						
2	0	0	0	1	1		r	27	6	7	5
6	3	2	1	0	0						
2	0	0	0	1	1		s	21	6	9	5
2	1	1	0	0	0	0					
2	0	0	1	1	0	0	r	27	8	7	6
2	0	0	0	0	1	1					

Table 2: IP CWS for $n = 3$. The columns indicate the minimality type ('s' for span, 'l' for lp-minimality and 'r' for r-minimality) and point and vertex numbers for Δ and Δ^*. As r-minimality implies lp-minimality and the latter implies the span property for $n = 3$, we have given only the strongest statement in each case.

d	n_1	n_2	n_3	n_4		P	V	$\overline{P}$	$\overline{V}$	Π	F
4	1	1	1	1	r	35	4	5	4	0	0
5	1	1	1	2	r	34	6	6	5	0	0
6	1	1	1	3	r	39	4	6	4	0	0
6	**1**	1	2	2	r	30	4	6	4	1	1
7	**1**	1	2	3	r	31	7	8	6	1	1
8	1	2	2	3	s	24	6	8	5	0	0
8	**1**	1	2	4	r	35	4	7	4	1	1
9	1	**2**	3	3	s	23	6	8	5	1	1
9	**1**	1	3	4	r	33	5	9	5	1	1
10	1	2	2	5	s	28	4	8	4	0	0
10	1	**2**	3	4	s	23	7	11	6	1	1
10	**1**	1	3	5	r	36	5	9	5	1	1
11	1	**2**	3	5	l	24	8	13	7	1	1
12	1	**2**	3	6	s	27	4	9	4	1	1
12	1	**2**	4	5	s	24	5	12	5	1	1
12	1	**3**	4	4	s	21	4	9	4	1	1
12	2	**3**	3	**4**	s	15	4	9	4	2	2
12	**1**	1	4	6	r	39	4	9	4	1	1
12	**2**	2	3	5	s	17	5	11	5	1	1
13	1	**3**	4	5	l	20	7	15	7	1	1
14	1	**2**	4	7	s	27	5	12	5	1	1
14	2	3	**4**	**5**	s	13	7	16	7	3	2
14	**2**	2	3	7	s	19	5	11	5	1	1
15	1	**2**	5	7	s	26	6	17	6	1	1
15	1	**3**	4	7	s	22	6	17	6	1	1
15	1	**3**	5	6	s	21	5	15	5	1	1
15	2	3	5	5	s	14	6	11	5	1	0
15	**3**	3	4	5	s	12	5	12	5	1	1
16	1	**2**	5	8	s	28	5	14	5	1	1
16	1	**3**	4	8	s	24	5	12	5	1	1
16	1	**4**	5	6	s	19	6	17	6	1	1
16	2	3	**4**	7	s	14	6	18	6	2	1
17	2	3	**5**	7	l	13	8	20	8	2	1
18	1	**2**	6	9	s	30	4	12	4	1	1
18	1	**3**	5	9	s	24	5	15	5	1	1
18	1	**4**	6	7	s	19	6	20	6	1	1
18	2	3	**4**	9	s	16	5	14	5	2	1
18	2	3	**5**	8	s	14	6	20	6	2	1
18	3	4	5	**6**	s	10	6	17	6	?	1
19	3	4	5	**7**	l	9	7	24	8	?	1
20	1	**4**	5	10	s	23	4	13	4	1	1
20	2	3	**5**	10	s	16	5	14	5	2	1
20	2	5	**6**	**7**	–	11	5	23	5	3	2
20	2	**4**	5	9	–	13	4	23	4	1	1
20	3	4	5	8	s	10	6	22	6	?	0
21	1	**3**	7	10	–	24	4	24	4	1	1
21	1	**5**	7	8	–	18	5	24	5	1	1
21	2	3	**7**	9	s	14	6	23	6	2	1

d	n_1	n_2	n_3	n_4		P	V	$\overline{P}$	$\overline{V}$	Π	F
21	3	5	**6**	7	–	9	5	21	5	?	1
22	1	**3**	7	11	–	25	5	20	5	1	1
22	1	**4**	6	11	s	22	6	20	6	1	1
22	2	**4**	5	11	–	14	5	19	5	1	1
24	1	**3**	8	12	s	27	4	15	4	1	1
24	1	**6**	8	9	s	18	5	24	5	1	1
24	2	3	8	11	–	15	4	27	4	1	0
24	2	3	**7**	12	s	16	5	20	5	2	1
24	3	4	5	12	s	12	5	18	5	?	0
24	3	4	**7**	10	s	10	5	26	6	2	1
24	3	**6**	7	8	–	9	4	21	4	?	1
24	4	5	6	**9**	s	8	5	26	6	?	1
25	4	5	7	**9**	–	7	5	32	6	?	1
26	1	**5**	7	13	–	21	5	24	5	1	1
26	2	3	8	13	–	16	5	23	5	1	0
26	2	5	**6**	13	–	13	5	23	5	2	1
27	2	5	**9**	11	–	11	6	32	6	2	1
27	5	6	7	9	–	6	5	30	6	?	0
28	1	**4**	9	14	–	24	4	24	4	1	1
28	3	4	**7**	14	s	12	5	18	5	2	1
28	4	6	7	**11**	–	7	4	35	4	?	1
30	1	**4**	10	15	s	25	5	20	5	1	1
30	1	**6**	8	15	s	21	5	24	5	1	1
30	2	3	10	15	s	18	4	18	4	1	0
30	2	**6**	7	15	–	13	4	23	4	1	1
30	3	4	**10**	13	–	10	5	35	5	2	1
30	4	5	6	15	s	10	5	20	5	?	0
30	5	6	8	**11**	–	6	4	39	4	?	1
32	2	5	**9**	16	–	13	5	29	5	2	1
32	4	5	7	16	–	9	5	27	5	?	0
33	3	5	**11**	14	–	9	4	39	4	2	1
34	3	4	**10**	17	–	11	6	31	6	2	1
34	4	6	7	**17**	–	8	5	31	5	?	1
36	1	**5**	12	18	–	24	4	24	4	1	1
36	3	4	**11**	18	–	12	4	30	4	2	1
36	7	8	9	12	–	5	4	35	4	?	0
38	3	5	**11**	19	–	10	5	35	5	2	1
38	5	6	8	**19**	–	7	5	35	5	?	1
40	5	7	8	20	–	8	4	28	4	?	0
42	1	**6**	14	21	s	24	4	24	4	1	1
42	2	5	14	21	–	15	4	27	4	1	0
42	3	4	**14**	21	s	13	5	26	5	2	1
44	4	5	**13**	22	–	9	4	39	4	2	1
48	3	5	**16**	24	–	12	4	30	4	2	1
50	7	8	10	**25**	–	6	4	39	4	?	1
54	4	5	**18**	27	–	10	5	35	5	2	1
66	5	6	**22**	33	–	9	4	39	4	2	1

Table 3: The 95 K3 weight systems: r, l, s denote the minimality type as in table 2, Π is the number of reflexive projections (if known) and F denotes the number of reflexive projections onto facets. The corresponding weights with unique partitions are indicated with bold face.

d	n_1	n_2	n_3	n_4	n_5	TM	h_{11}	h_{12}	P	V	$\overline{P}$	$\overline{V}$	Π	F
47	3	4	5	14	21	−ls	26	39	54	18	35	15	?	0
69	7	8	10	19	25	−ls	59	10	16	13	75	21	?	0
97	7	8	11	**26**	45	−ls	63	15	24	15	71	21	?	1
84	**1**	1	12	28	42	T r	11	491	680	5	26	5	1	1
280	7	19	**40**	87	127	−−	491	11	26	5	680	5	2	1
24	3	4	5	6	6	T s	10	34	36	8	12	7	?	0
26	3	4	5	7	7	−ls	22	22	31	13	21	10	?	0
33	3	6	6	7	11	−−	19	37	34	7	22	6	?	0
36	3	6	6	10	11	T−	19	49	38	7	22	6	?	0
26	3	4	5	**6**	8	− s	14	24	32	14	19	10	?	1
36	5	**7**	7	8	9	−−	30	12	19	10	28	9	?	1
39	3	**6**	9	10	11	T s	17	41	33	12	22	13	?	1
52	4	6	**8**	11	23	T−	29	33	34	9	36	8	?	1
34	3	**6**	7	8	**10**	− s	18	20	27	13	23	12	?	2
44	4	**8**	9	10	**13**	−−	29	17	22	9	31	9	?	2
55	3	10	**13**	14	**15**	T s	28	16	23	12	35	14	?	2
63	7	9	**14**	15	**18**	T−	44	8	15	6	37	6	?	2
5	1	1	1	1	1	T r	1	101	126	5	6	5	0	0
10	1	1	1	3	4	− r	4	126	165	10	9	7	0	0
25	1	5	5	6	8	T−	17	49	65	7	15	7	0	0
26	1	5	5	7	8	− l	19	49	65	9	19	7	0	0
20	2	3	4	4	7	− s	13	45	51	10	14	8	1	0
20	2	3	5	5	5	T s	6	48	50	8	11	6	1	0
30	2	5	6	6	11	−−	27	39	45	7	25	6	1	0
36	2	5	6	6	17	T−	24	54	60	7	25	6	1	0
8	**1**	1	2	2	2	T r	2	86	105	5	7	5	1	1
13	**1**	1	2	4	5	− r	6	108	141	12	11	8	1	1
35	2	7	**8**	9	9	−−	35	23	33	11	30	8	1	1
40	4	5	**9**	10	12	T−	22	18	25	7	20	7	1	1
19	2	3	**4**	5	5	−ls	11	33	43	14	14	9	2	1
27	2	3	**4**	9	9	T s	14	44	56	9	13	7	2	1
36	**4**	4	6	9	13	−−	31	31	33	6	29	6	2	1
40	**4**	4	6	9	17	T−	26	38	39	7	29	6	2	1
14	**2**	2	**3**	3	4	T r	5	51	57	10	10	7	2	2
19	2	**3**	3	**4**	7	−ls	11	39	51	14	16	9	2	2
30	3	**5**	5	**6**	11	−−	33	21	33	6	25	6	2	2
35	3	**5**	5	**6**	16	T−	26	28	42	7	25	6	2	2
28	4	**5**	5	6	**8**	− s	18	20	27	10	18	8	3	2
36	4	6	**8**	**9**	9	T s	23	23	26	6	16	6	3	2
40	4	**7**	7	10	**12**	−−	28	16	23	7	25	6	3	2
42	6	**7**	7	10	**12**	T−	35	11	19	6	23	6	3	2

Table 4: Examples from our list of 184026 IP weights [33] with various data including Hodge numbers, point and vertex numbers, and numbers of reflexive projections (onto facets). T indicates transversality and M denotes the minimality type.

References

[1] L. Dixon, in *Superstrings, unified theories, and cosmology 1987,* eds. G.Furlan et al. (World Scientific, 1988) p. 67.

[2] W. Lerche, C. Vafa, N. Warner, *Chiral rings in N=2 superconformal theories,* Nucl. Phys. **B324** (1989) 427.

[3] P. Candelas, E. Derrick, L. Parkes, *Generalized Calabi-Yau manifolds and the mirror of a rigid manifold,* Nucl. Phys. **B407** (1993) 115, hep-th/9304045.

[4] P.S. Aspinwall, B.R. Greene, *On the geometric interpretation of $N = 2$ superconformal theories,* Nucl. Phys. **B437** (1995) 205, hep-th/9409110.

[5] P. Candelas, X.C. De la Ossa, P.S. Green, L. Parkes, *An Exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds,* Phys. Lett. **B258** (1991) 118.

[6] P. Candelas, X.C. De la Ossa, P.S. Green, L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory,* Nucl. Phys. **B359** (1991) 21.

[7] A. Strominger, S.-T. Yau, E. Zaslow, *Mirror symmetry is T duality,* Nucl. Phys. **B479** (1996) 243, hep-th/9606040.

[8] P. Candelas, A.M. Dale, C.A. Lutken, R. Schimmrigk, *Complete intersection Calabi-Yau manifolds,* Nucl. Phys. **B298** (1988) 493.

[9] P. Candelas, M. Lynker and R. Schimmrigk, *Calabi-Yau manifolds in weighted $\mathbb{P}_4$,* Nucl. Phys. **B341** (1990) 383.

[10] M. Kreuzer, H. Skarke, *No mirror symmetry in Landau-Ginzburg spectra!,* Nucl. Phys. **B388** (1992) 113, hep-th/9205004.

[11] A. Klemm, R. Schimmrigk, *Landau–Ginzburg string vacua,* Nucl. Phys. **B411** (1994) 559, hep-th/9204060.

[12] M. Kreuzer, H. Skarke, *All abelian symmetries of Landau–Ginzburg potentials,* Nucl. Phys. **B405** (1993) 305, hep-th/9211047.

[13] P. Berglund, T. Hübsch, *A generalized construction of mirror manifolds,* Nucl. Phys. **B393** (1993) 377, hep-th/9201014.

[14] M. Kreuzer, H. Skarke, *Orbifolds with discrete torsion and mirror symmetry,* Phys. Lett. **B357** (1995) 81, hep-th/9505120.

[15] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties,* J. Alg. Geom. **3** (1994) 493, alg-geom/9310003.

[16] P.S. Aspinwall, B.R. Greene, D.R. Morrison, *Space-time topology change and stringy geometry,* J. Math. Phys. **35** (1994) 5321, hep-th/9311186.

[17] P. Candelas, P.S. Green, T. Hübsch, *Rolling among Calabi-Yau vacua,* Nucl. Phys. **B330** (1990) 49.

[18] *Finite distances between distinct Calabi-Yau vacua: (Other worlds are just around the corner),* Phys. Rev. Lett. **62** (1989) 1956.

[19] A. Strominger, *Massless black holes and conifolds in string theory,* Nucl. Phys. **B451** (1995) 96, hep-th/9504090.

[20] B.R. Greene, D.R. Morrison, A. Strominger, *Black hole condensation and the unification of string vacua,* Nucl. Phys. **B451** (1995) 109, hep-th/9504145.

[21] S. Kachru, C.Vafa, *Exact results for $N = 2$ compactifications of heterotic strings,* Nucl. Phys. **B450** (1995) 69, hep-th/9505105.

[22] C. Vafa, *Evidence for F-theory,* Nucl. Phys. **B469** (1996) 403, hep-th/9602022.

[23] M. Kreuzer, H.Skarke, *On the classification of reflexive polyhedra,* Commun. Math. Phys. **185** (1997) 495, hep-th/9512204.

[24] M.Kreuzer, H. Skarke, *On the classification of quasihomogeneous functions,* Commun. Math. Phys. **150** (1992) 137, hep-th/9202039.

[25] P. Candelas, X.C. de la Ossa, S. Katz, *Mirror symmetry for Calabi–Yau hypersurfaces in weighted $\mathbb{P}^4$ and extensions of Landau–Ginzburg theory,* Nucl. Phys. **B450** (1995) 267, hep-th/9412117.

[26] H. Skarke, *Weight Systems for toric Calabi–Yau varieties and reflexivity of Newton polyhedra,* Mod. Phys. Lett. **A11** (1996) 1637, alg-geom/9603007.

[27] A. Klemm, W. Lerche and P. Mayr, *K3–fibrations and heterotic-type II string duality,* Phys. Lett. **B357** (1995) 313, hep-th/9506112.

[28] P.S. Aspinwall and J. Louis, *On the ubiquity of $K3$ fibrations in string duality,* Phys. Lett. **B369** (1996) 233, hep-th/9510234.

[29] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi–Yau threefolds – I,* Nucl. Phys. **B473** (1996) 74, hep-th/9602114.

[30] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi–Yau threefolds – II,* Nucl. Phys. **B476** (1996) 437, hep-th/9603161.

[31] A. Avram, M. Kreuzer, M. Mandelberg, H. Skarke, *Searching for $K3$ fibrations,* Nucl. Phys. **B494** (1997) 567, hep-th/9610154.

[32] M. Kreuzer, H. Skarke, *Calabi–Yau fourfolds and toric fibrations,* J. Geom. Phys. **466** (1997) 1, hep-th/9701175.

[33] M. Kreuzer, H. Skarke, `http://hep.itp.tuwien.ac.at/~kreuzer/CY.html`

[34] C. Vafa, *String vacua and orbifoldized LG models,* Mod. Phys. Lett. **A4A** (1989) 1169; Superstring Vacua, HUTP-89/A057 preprint.

[35] V.V. Batyrev, D.I. Dais, *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry,* Topology **35** (1996) 901, alg-geom/9410001.

[36] M. Kreuzer, H. Skarke, *Landau–Ginzburg orbifolds with discrete torsion,* Mod. Phys. Lett. **A10** (1995) 1073, hep-th/9412033.

[37] C. Vafa, *Modular invariance and discrete torsion on orbifolds,* Nucl. Phys. **B273** (1986) 592.

[38] A. Klemm, `http://ars-www.uchicago.edu/~aklemm/HTML/cy.htm`

[39] V.V. Batyrev, L.A. Borisov, *Dual cones and mirror symmetry for generalized Calabi–Yau manifolds,* in *Mirror Symmetry II* (ed. S.-T. Yau), alg-geom/9402002.

[40] V.V. Batyrev, L.A. Borisov, *On Calabi-Yau complete intersections in toric varieties,* in *Proceedings of Trento Conference (1994)*, alg-geom/9412017.

[41] V.V. Batyrev, L.A. Borisov, *Mirror duality and string theoretic Hodge numbers,* alg-geom/9509009.

[42] R. Schimmrigk, `http://thew02.physik.uni-bonn.de/~netah/cy.html`

[43] S. Katz, `http://www.math.okstate.edu/~katz/CY/`

[44] M. Reid, *Canonical 3-folds,* Proc. Alg. Geom. Anger 1979, Sijthoff and Nordhoff, 273.

[45] A.R. Fletcher, *Working with complete intersections,* Bonn preprint MPI/89–35 (1989).

[46] V.V. Batyrev, *Higher-dimensional toric varieties with ample anticanonical class,* Moscow State Univ., Thesis, 1985 (Russian).

[47] R.J. Koelman, *The number of moduli of families of curves on toric varieties,* Katholieke Universiteit Nijmegen, Thesis, 1990.

[48] W. Fulton, *Introduction to toric varieties* (Princeton Univ. Press, Princeton 1993).

[49] T. Oda, *Convex bodies and algebraic geometry* (Springer, Berlin Heidelberg 1988).

[50] D. Cox, *The Homogeneous coordinate ring of a toric variety,* J. Alg. Geom. **4** (1995) 17, alg-geom/9210008.

[51] Igor V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces,* alg-geom/9502005.

[52] P.S. Aspinwall, $K3$ *surfaces and string duality,* hep-th/9611137.

[53] P. Candelas, H. Skarke, *F-theory,* $SO(32)$ *and toric geometry,* Phys. Lett. **B413** (1997) 63, hep-th/9706226.

[54] E.Witten, *String theory dynamics in various dimensions,* hep-th/9503124, Nucl. Phys. **B443** (1995) 85.

[55] P. Candelas, A. Font, *Duality between the webs of heterotic and type II vacua,* Nucl. Phys. **B511** (1998) 295, hep-th/9603170.

[56] A.Klemm, W. Lerche, P.Mayr, *K3–Fibrations and heterotic-type II string duality,* Phys. Lett. **B357** (1995) 313, hep-th/9506112.

[57] S. Hosono, B.H. Lian, S.-T. Yau, *Calabi–Yau varieties and pencils of K3 surfaces,* alg-geom/9603020.